

# APPLICATIONS OF THE CHANGE-OF-RINGS SPECTRAL SEQUENCE TO THE COMPUTATION OF HOCHSCHILD COHOMOLOGY

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*Dedicated to Henri Cartan and Samuel Eilenberg.*

**ABSTRACT.** We consider the change-of-rings spectral sequence as it applies to Hochschild cohomology, obtaining a description of the differentials on the first page which relates it to the multiplicative structure on cohomology. Using this information, we are able to completely describe the cohomology structure of monogenic algebras as well as some information on the structure of the cohomology in more general situations.

We also show how to use the spectral sequence to reprove and generalize results of M. Auslander *et al.* about homological epimorphisms. We derive from this a rather general version of the long exact sequence due to D. Happel for a one-point (co)-extension of a finite dimensional algebra and show how it can be put to use in concrete examples.

## INTRODUCTION

The computation of the Hochschild cohomology  $HH^\bullet(A)$  of an algebra  $A$  is usually a difficult, laborious task. In most cases, the well-known identification  $HH^\bullet(A) \cong \text{Ext}_{A^e}^\bullet(A, A)$  and the method developed by Cartan and Eilenberg in *Homological Algebra* [5] for the computation of derived functors are used: one finds an  $A^e$ -projective resolution  $P^\bullet$  for  $A$  and then computes the cohomology of the complex  $\text{hom}_{A^e}(P^\bullet, A)$ . Up to the flexibility of being able to substitute the standard Hochschild resolution for a more convenient one, this method is a direct implementation of the definition of  $HH^\bullet(A)$  given by Hochschild in [20].

As it is well known, the Hochschild cohomology  $HH^\bullet(A)$  is endowed naturally with both an associative algebra structure and a graded Lie algebra structure. Recently, these structures have received considerable attention: the rôle played by the former in the representation theory of  $A$  is being studied by many researchers and a theory modeled on Quillen's theory of support varieties for groups [27] is emerging, while the latter has been central in recent developments related to the deformation theory of algebras.

Now, as soon as one attempts to make these structures explicit in specific examples, computational difficulties arise very quickly. The cup product can be computed using an arbitrary  $A^e$ -projective resolution  $P^\bullet$  of  $A$  provided with a diagonal map  $\Delta : P^\bullet \rightarrow P^\bullet \otimes_A P^\bullet$  but, while this is in general considerably more convenient than dealing with the standard Hochschild resolution, this quickly becomes impractical. The Lie structure, on the other hand, is defined in terms of the Hochschild resolution and we do not have available—to the author's knowledge—a way of computing it in terms of an arbitrary resolution. Consequently,

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when a projective  $A^\ell$ -resolution  $P^\bullet$  of  $A$  is being used to determine  $HH^\bullet(A)$ , in order to compute the Lie structure one needs comparison maps  $P^\bullet \rightrightarrows B^\bullet(A)$  between  $P^\bullet$  and the Hochschild resolution  $B^\bullet(A)$ . This is extremely messy; in most cases, in fact, this approach is used successfully only in low degrees.

This rather unsatisfactory situation should be compared to what happens in the context of group cohomology. Indeed, while by no means a trivial task, the effective computation of group cohomology as an algebra is a rather well understood process in which the various pieces of structure present—the Lyndon-Hochschild-Serre spectral sequence, the action of the Steenrod algebra and, more generally, the close relationship with algebraic topology, etc.—serve as powerful tools. At present, we do not have comparable tools at hand when dealing with Hochschild cohomology.

In this paper, we pick one of the general constructions of homological algebra, the spectral sequence for a change of rings, and we try to put it to use in the problem of computing Hochschild cohomology. It turns out that the differentials on the initial term of this spectral sequence can be described in terms of the Yoneda product. Our main technical result, and the objective of sections 1 and 2 below, is the following theorem:

**Theorem.** *Let  $k$  be a field and let  $\phi : A \rightarrow B$  be a morphism of  $k$ -algebras. If  $M$  is a  $B$ -bimodule, there is a natural convergent spectral sequence such that*

$$E_2^{p,q} \cong \text{Ext}_{B^e}^p(\text{Tor}_q^A(B, B), M) \Rightarrow H^\bullet(A, M).$$

Moreover, for each  $q \geq 1$  there exists  $\mathcal{O}_q(\phi) \in \text{Ext}_{B^e}^2(\text{Tor}_{q-1}^A(B, B), \text{Tor}_q^A(B, B))$  such that the differential on the term  $E_2$ ,

$$d_2^{p,q} : \text{Ext}_{B^e}^p(\text{Tor}_q^A(B, B), M) \rightarrow \text{Ext}_{B^e}^{p+2}(\text{Tor}_{q-1}^A(B, B), M),$$

is given by

$$d_2^{p,q}(\zeta) = (-1)^p \zeta \circ \mathcal{O}_q(\phi)$$

if  $\zeta \in \text{Ext}_{B^e}^p(\text{Tor}_q^A(B, B), M)$  and  $\circ$  is the Yoneda product.

We remark that this theorem provides invariants  $\mathcal{O}_q(\phi)$  for the morphism  $\phi$ . When  $\phi$  is surjective,  $\mathcal{O}_1(\phi)$  is well-known:

**Proposition.** *Assume, in the theorem, that  $\phi : A \rightarrow B$  is surjective and put  $I = \ker \phi$ . Then the class  $\mathcal{O}_1(\phi)$  can be seen as an element of  $H^2(B, I/I^2)$  and then coincides with the characteristic class of the singular extension of algebras*

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0$$

The classes  $\mathcal{O}_q(\phi)$  can be considered, then, as higher degree analogues of the characteristic class. It would be interesting to have tractable descriptions for them and their possible interrelations.

We remark that we have not been able to obtain information on the differentials  $d^p$  for  $p > 2$  of the spectral sequence appearing in the theorem. Such information would be quite useful.

Using our theorem, in section 3, we are able to work out a computation of the Hochschild cohomology of algebras  $k[X]/(f)$  which are quotients of a polynomial algebra in one variable, arriving at a presentation of the cohomology algebra  $HH^\bullet(k[X]/(f))$  and of the Gerstenhaber Lie bracket on it; see theorems 3.9 and 3.11 below for precise statements. Interestingly, this computation relies on explicit work with resolutions only in very low degrees. The sort of arguments used for this are of wider applicability: in section 4 we present some variations and we intend, in future work, to provide further related results.

Finally, in section 5 we consider the change-of-rings spectral sequence corresponding to an homological epimorphism  $\phi : A \rightarrow B$ , i.e., a morphism of algebras which induces a full and faithful embedding  $D^b({}_B\text{Mod}) \rightarrow D^b({}_A\text{Mod})$  of bounded derived categories. We are able to generalize results which relate the Hochschild cohomologies of  $A$  and  $B$  due to M. Auslander, M. I. Platzeck and G. Todorov [1] and J. A. de la Peña and C. Xi [26]. In particular, following [26] we obtain long exact sequences generalizing the one constructed by D. Happel in [17] for one-point (co)extensions of finite dimensional algebras as well as the generalizations of C. Cibils [6], E. Green and Ø. Solberg [15], E. Green, E. N. Marcos and N. Snashall [16].

**Global conventions.** We fix a field  $k$  throughout the paper. Algebras will be always be  $k$ -algebras,  $\text{hom}$  and  $\otimes$  will be taken over  $k$  and, in general, all our linear constructions will be  $k$ -linear.

When working with algebras given as admissible quotients of path algebras, our notations and nomenclature should be standard. As we prefer left modules, we compose arrows from right to left. We consider paths in a quiver as elements both of the path algebra, as usual, and of the quotient algebras thereof, provided they represent a non-zero element in such a quotient.

We refer to Stanley's book [28, Chapter 3] for the little we need about partially ordered sets.

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## 1. STABLE OPERATIONS ON COHOMOLOGY

**1.1.** Let  $A$  be an algebra. Recall that composition of extensions gives, for left  $A$ -modules  $M, N$  and  $P$ , a Yoneda product

$$\circ : \text{Ext}_A^\bullet(N, P) \otimes \text{Ext}_A^\bullet(M, N) \rightarrow \text{Ext}_A^\bullet(M, P)$$

which is homogeneous, natural and associative. Given a short exact sequence  $E : N' \rightarrowtail N \twoheadrightarrow N''$  of left  $A$ -modules and a left  $A$ -module  $M$ , the connecting homomorphisms in the long exact sequence for the  $\partial$ -functor  $\text{Ext}_A^\bullet(M, -)$ , namely  $\partial : \text{Ext}_A^q(M, N'') \rightarrow \text{Ext}_A^{q+1}(M, N')$ , are given, for  $\sigma \in \text{Ext}_A^q(M, N'')$ , by the simple formula

$$\partial\sigma = E \circ \sigma,$$

where we regard  $E \in \text{Ext}_A^1(N'', N')$ . This is explained in detail in [24], chapter III.

**1.2.** Given a pair of functors  $F, T : A \rightarrow B$  between categories  $A$  and  $B$ , we write  $[F, T]$  the class of natural transformations  $F \rightarrow T$ . If  $f \in [F, T]$  and  $x \in A$ , we write  $f_x : Fx \rightarrow Tx$  the component morphism of  $B$  corresponding to  $x$ .

**1.3.** Fix now an algebra  $A$ . If  $p, q \geq 0$  and  $M$  and  $N$  are left  $A$ -modules, set

$$\text{Op}_A^{p,q}(M, N) = [\text{Ext}_A^p(N, -), \text{Ext}_A^q(M, -)].$$

Clearly  $\text{Op}_A^{p,q}$  is an additive functor, contravariant on its first variable, covariant on the second one. Yoneda's lemma gives us an isomorphism of functors

$$\text{Op}_A^{0,q}(M, N) = [\text{hom}_A(N, -), \text{Ext}_A^q(M, -)] \cong \text{Ext}_A^q(M, N).$$

**1.4.** Given left  $A$ -modules  $M$  and  $N$  and  $d \geq 0$ , a *cohomological operation*  $\mathcal{O}$  of degree  $d$  from  $M$  to  $N$  is a sequence  $\mathcal{O} = (\mathcal{O}^p)_{p \geq 0}$  of natural transformations  $\mathcal{O}^p \in \text{Op}_A^{p,p+d}(M, N)$ . We say that  $\mathcal{O}$  is *stable* if, for each short exact sequence  $P' \hookrightarrow P \twoheadrightarrow P''$  of left  $A$ -modules, the following diagram commutes for each  $p \geq 0$ :

$$\begin{array}{ccc} \text{Ext}_A^p(N, P'') & \xrightarrow{\partial} & \text{Ext}_A^{p+1}(N, P') \\ \mathcal{O}_{P''}^p \downarrow & & \downarrow \mathcal{O}_{P'}^{p+1} \\ \text{Ext}_A^{p+d}(M, P'') & \xrightarrow{\partial} & \text{Ext}_A^{p+1+d}(M, P') \end{array}$$

We write  $\text{sOp}_A^d(M, N)$  the class of all stable cohomological operations from  $M$  to  $N$ . Again, it is clear that this is an additive functor, with the same variances as  $\text{Op}_A^{\bullet, \bullet}(-, -)$ .

**1.5.** The Yoneda product allows us to define, for each  $d \geq 0$ , a natural morphism  $Y : \text{Ext}_A^d(M, N) \rightarrow \text{sOp}_A^d(M, N)$  in the following way: if  $\zeta \in \text{Ext}_A^d(M, N)$ ,  $p \geq 0$  and  $P$  is a left  $A$ -module, we let  $Y(\zeta)_P^p = (-) \circ \zeta : \text{Ext}_A^p(N, P) \rightarrow \text{Ext}_A^{p+d}(M, P)$  be given by post-multiplication by  $\zeta$ . This is well defined because  $Y(\zeta)$  is a cohomological operation by definition, which is stable because the Yoneda product is associative and the connecting homomorphisms can be expressed as in 1.1. The morphism  $Y$  is additive and natural both in  $M$  and  $N$ .

It is a monomorphism; in fact, writing  $1_N \in \text{hom}_A(N, N) = \text{Ext}_A^0(N, N)$  the identity map of  $N$ , if  $\zeta \in \text{Ext}_A^d(M, N)$ , then  $Y(\zeta)_N^0(1_N) = \zeta$ , so we recover  $\zeta$  from  $Y(\zeta)$ . On the other hand, if  $\mathcal{O} \in \text{sOp}_A^d(M, N)$  and  $E_{\mathcal{O}} = \mathcal{O}_N^0(1_N) \in \text{Ext}_A^d(M, N)$ , put  $\mathcal{R} = Y(E_{\mathcal{O}}) - \mathcal{O}$ ; this is a stable cohomological operation of degree  $d$ . From the very definition,  $\mathcal{R}_N^0(1_N) = 0$ ; if  $f \in \text{hom}_A(N, P) = \text{Ext}_A^0(N, P)$  and, for each left  $A$ -module  $Q$ ,  $f_Q^q : \text{Ext}_A^q(Q, N) \rightarrow \text{Ext}_A^q(Q, P)$  is the induced morphism, naturality implies that  $\mathcal{R}_P^0(f) = \mathcal{R}_P^0 f_N^0(1_N) = f_M^q \mathcal{R}_N^0(1_N) = 0$ . This means that in fact  $\mathcal{R}^0 = 0$ . We will show in 1.7 that any stable cohomological operation which vanishes—as  $\mathcal{R}$  does—on  $\text{Ext}_A^0$  is identically zero; this will allow us to conclude that  $Y(E_{\mathcal{O}}) = \mathcal{O}$ , proving the following theorem:

**1.6. Theorem.** *There is an isomorphism of graded bifunctors*

$$Y : \text{Ext}_A^{\bullet}(-, -) \xrightarrow{\cong} \text{sOp}_A^{\bullet}(-, -).$$

**1.7.** Consider, then, a stable cohomological operation  $\mathcal{O} \in \text{sOp}_A^d(M, N)$  and suppose that  $p \geq 0$  and  $\mathcal{O}^p = 0$ . Let  $P$  be a left  $A$ -module and choose any short exact sequence  $P \hookrightarrow I \twoheadrightarrow P'$  in which  $I$  is an injective module. Stability of  $\mathcal{O}$  entails the commutation of the following diagram, in which the top row is exact:

$$\begin{array}{ccccc} \text{Ext}_A^p(N, P') & \xrightarrow{\partial} & \text{Ext}_A^{p+1}(N, P) & \longrightarrow & \text{Ext}_A^{p+1}(N, I) = 0 \\ \mathcal{O}_{P'}^p \downarrow & & \downarrow \mathcal{O}_P^{p+1} & & \\ \text{Ext}_A^{p+d}(M, P') & \xrightarrow{\partial} & \text{Ext}_A^{p+1+d}(M, P) & \longrightarrow & \text{Ext}_A^{p+1+d}(M, I) = 0 \end{array}$$

The hypothesis that  $\mathcal{O}^p = 0$  implies that  $\mathcal{O}_P^{p+1} = 0$ . The arbitrariness of  $P$  and an evident inductive argument show that  $\mathcal{O} = 0$ , as we needed.

**1.8.** Our definition of stable cohomological operation restricts us to consider operations of non-negative degree. The proof of theorem 1.6 shows that there is no loss in this, since an operation of negative degree will certainly vanish on  $\text{Ext}_A^0$ . We note however that in general one has  $\text{Op}_A^{p,q}(M, N) \neq 0$  for  $p > q$ , cf. [18].

**1.9.** It is clear that under the isomorphism of the theorem, the Yoneda product is identified with the composition of stable operations.

## 2. CHANGE OF RINGS

### 2.1. The spectral sequence.

**2.1.1.** Consider a morphism of algebras  $\phi : A \rightarrow B$  and left  $A$ - and  $B$ -modules  $M$  and  $N$ , respectively. Let  $X^\bullet \rightarrow M$  be a projective resolution of  $M$  as an  $A$ -module and  $N \hookrightarrow Y^\bullet$  an injective resolution of  $N$  as a  $B$ -module. We consider the complex  $Z^\bullet = \text{hom}_B(B \otimes_A X^\bullet, Y^\bullet)$ .

**2.1.2.** The filtration  $'F^\bullet Z^\bullet$  on  $Z^\bullet$  with

$$'F^p Z^q = \bigoplus_{\substack{r+s=q \\ r \geq p}} \text{hom}_B(B \otimes_A X^r, Y^s)$$

determines a spectral sequence  $'E$  converging to  $H(Z^\bullet)$ . The differential on  $'E_0$  is induced by the one on  $Y^\bullet$  and, since there is an isomorphism of functors  $\text{hom}_B(B \otimes_A -, -) \cong \text{hom}_A(-, -)$  and each  $X^p$ , when  $p \geq 0$ , is a projective  $A$ -module, we have that  $'E_1^{p,q} = 0$  if  $q > 0$  and  $'E_1^{\bullet,0} \cong \text{hom}_A(X^\bullet, N)$ . The differential on  $'E_1$  corresponds in a natural way to the differential on  $X^\bullet$ , so that  $'E_2^{\bullet,0} \cong \text{Ext}_A^\bullet(M, N)$ . The spectral sequence degenerates at the term  $'E_2$  and convergence implies that  $H(Z^\bullet) \cong \text{Ext}_A^\bullet(M, N)$ .

**2.1.3.** Considering now the filtration  $''F^\bullet Z^\bullet$  given by

$$''F^q Z^p = \bigoplus_{\substack{r+s=p \\ s \geq q}} \text{hom}_B(B \otimes_A X^r, Y^s)$$

we obtain another cohomologically graded spectral sequence contained in the first quadrant converging to  $\text{Ext}_A^\bullet(M, N)$ . We have  $''E_0^{p,q} \cong \text{hom}_B(B \otimes_A X^q, Y^p)$  with differential induced by the one on  $X^\bullet$ ; as  $\text{hom}_B(-, Y^p)$  is an exact functor for each  $p \geq 0$ , we find that  $''E_1^{p,q} \cong \text{hom}_B(\text{Tor}_q^A(B, M), Y^p)$ . The differential on the term  $''E_1$  is induced by the differential on  $Y^\bullet$ , so that the next term has  $''E_2^{p,q} \cong \text{Ext}_B^p(\text{Tor}_q^A(B, M), N)$ .

We record these facts in the following proposition.

**2.1.4. Proposition.** *Let  $\phi : A \rightarrow B$  be a morphism of algebras. Let  $M$  be a left  $A$ -module and  $N$  a left  $B$ -module and consider  $N$  as a left  $A$ -module by pull-back along  $\phi$ . There is a convergent spectral sequence*

$$E_2^{p,q} \cong \text{Ext}_B^p(\text{Tor}_q^A(B, M), N) \Rightarrow \text{Ext}_A^\bullet(M, N).$$

*It is natural with respect to both  $M$  and  $N$ .* □

**2.1.5.** This spectral sequence was first constructed in [5, XVI.5, case 3].

**2.1.6.** Standard properties of resolutions and the comparison theorem for spectral sequences immediately imply that the spectral sequence of 2.1.4 does not depend on the particular resolutions  $X^\bullet$  and  $Y^\bullet$  used to construct them.

**2.1.7.** The spectral sequence constructed in 2.1 is in the first quadrant, so it comes equipped with an edge morphism from the limit to the “base,”

$$e : \text{Ext}_A^\bullet(M, N) \rightarrow E_2^{0, \bullet} \cong \text{hom}_B(\text{Tor}_\bullet^A(B, M), N)$$

and a morphism from the “fiber” to the limit,

$$e' : E_2^{\bullet, 0} \cong \text{Ext}_B^\bullet(B \otimes_A M, N) \rightarrow \text{Ext}_A^\bullet(M, N). \quad (1)$$

They can be computed as follows.

Let  $X^\bullet \twoheadrightarrow M$  be a projective resolution of  $M$  as a left  $A$ -module and take a class  $\alpha \in \text{Ext}_A^p(M, N)$ . To compute  $e(\alpha)$ , we pick a representative  $a \in \text{hom}_A(X^p, N)$  for  $\alpha$  and let  $\bar{a} \in \text{hom}_B(B \otimes_A X^p, N)$  be the morphism corresponding to it by the natural identification, so that  $\bar{a}(b \otimes x) = ba(x)$ . If  $\tau \in \text{Tor}_p^A(B, M)$  is represented by  $t \in B \otimes_A X^p$  in the complex  $B \otimes_A X^\bullet$ , then

$$e(\alpha)(\tau) = \bar{a}(t).$$

Consider now additionally a projective resolution  $X'^\bullet \twoheadrightarrow B \otimes_A M$  of  $B \otimes_A M$  as a left  $B$ -module. There exists a morphism of complexes of left  $B$ -modules  $f : B \otimes_A X^\bullet \rightarrow X'^\bullet$  over the identity map of  $B \otimes_A M$ , well determined up to homotopy; indeed, the graded components of  $B \otimes_A X^\bullet$  are  $B$ -projective. The morphism  $e'$  is then induced on homology by the composition

$$\text{hom}_B(X'^\bullet, N) \xrightarrow{f^*} \text{hom}_B(B \otimes_A X^\bullet, N) \cong \text{hom}_A(X^\bullet, N).$$

**2.1.8.** Suppose that  $\phi : A \rightarrow B$  is a surjection of algebras. Then, of course, there is a canonical isomorphism  $B \otimes_A M \cong M$  if  $M \in {}_B\text{Mod}$  and the edge morphism from the fiber to the limit in the spectral sequence is, when both  $M$  and  $N$  are left  $B$ -modules,

$$e' : \text{Ext}_B^\bullet(M, N) \rightarrow \text{Ext}_A^\bullet(M, N).$$

In this situation, we can describe  $e'$  in terms of iterated extensions of modules: if  $\alpha \in \text{Ext}_B^p(M, N)$  is the class of the  $p$ -extension of left  $B$ -modules

$$0 \longrightarrow N \longrightarrow U_p \longrightarrow \cdots \longrightarrow U_1 \longrightarrow M \longrightarrow 0 \quad (2)$$

then  $e'(\alpha) \in \text{Ext}_A^p(M, N)$  is simply the  $p$ -extension (2) now considered in  ${}_A\text{Mod}$ . This follows easily from the recipe given in 2.1.7 for computing  $e'$  and the details of the proof that the Ext-functors can be computed from iterated extensions as presented in [19, Theorem IV.9.1]. We leave this to the reader and just recall briefly how one goes from iterated extensions to cocycles, as we will need this below.

Let  $M \in {}_A\text{Mod}$  and fix a projective resolution  $X_\bullet \rightarrow M$  of  $M$  in  ${}_A\text{Mod}$ . Consider a  $p$ -extension  $\xi$  in that category ending in  $M$ , as in the bottom row of the following diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & X_p & \longrightarrow & X_{p-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow h_p & & \downarrow h_{p-1} & & & & \downarrow h_0 & & \downarrow h_0 & & \downarrow \text{id}_M & & \\ \xi : & 0 & \longrightarrow & N & \longrightarrow & U_p & \longrightarrow & \cdots & \longrightarrow & U_2 & \longrightarrow & U_1 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since the top row is made of projective modules and the bottom one is exact, the identity map  $\text{id}_M$  on the right extends to a map of complexes  $h_\bullet$ . Clearly  $h_p : X_p \rightarrow N$  is actually a  $p$ -cocycle in  $\text{hom}_A(X_\bullet, N)$ ;  $h_\bullet$  is defined only up to homotopy and this means that  $h_p$  is defined up to a coboundary in  $\text{hom}_A(X_\bullet, N)$ . The class of  $h_p$  in  $H(\text{hom}_A(X_\bullet, N))$  depends thus only on  $\xi$  and it is its characteristic class in  $\text{Ext}_A^p(M, N)$ .

**2.1.9.** This description 2.1.8 of the morphism (1) in terms of extensions has the immediate consequence that  $e'$  is multiplicative on  $B$ -modules: if  $M, N, P \in {}_B\text{Mod}$  and  $\alpha \in \text{Ext}_B^p(M, N)$ ,  $\beta \in \text{Ext}_B^q(N, P)$ , then

$$e'(\beta \circ \alpha) = e'(\beta) \circ e'(\alpha),$$

if  $\circ$  denotes Yoneda composition of iterated extensions.

## 2.2. The differentials.

**2.2.1.** In the following lemma, we consider double complexes  $X^{\bullet, \bullet}$  whose horizontal and vertical differentials  $\delta'$  and  $\delta''$  anti-commute and cohomologically graded spectral sequences  $E^{\bullet, \bullet}$  for which the first upper index corresponds to the filtration degree.

**Lemma.** *Let*

$$0 \longrightarrow {}_1X^{\bullet, \bullet} \xrightarrow{j_0} {}_2X^{\bullet, \bullet} \xrightarrow{k_0} {}_3X^{\bullet, \bullet} \longrightarrow 0$$

*be a short exact sequence of double complexes such that the induced sequence*

$$0 \longrightarrow {}_1E_1^{\bullet, \bullet} \xrightarrow{j_1} {}_2E_1^{\bullet, \bullet} \xrightarrow{k_1} {}_3E_1^{\bullet, \bullet} \longrightarrow 0$$

*is also exact. If  $\partial : {}_3E_2^{p, q} \rightarrow {}_1E_2^{p, q+1}$  is the connecting homomorphism corresponding to the differentials in this last sequence, the square*

$$\begin{array}{ccc} {}_3E_2^{p, q} & \xrightarrow{\partial} & {}_1E_2^{p, q+1} \\ d_2 \downarrow & & \downarrow d_2 \\ {}_3E_2^{p-1, q+2} & \xrightarrow{\partial} & {}_1E_2^{p-1, q+3} \end{array}$$

*anti-commutes.*

The diagram in figure 1 may be of help in following the proof of the lemma. The dotted lines are used to show the relative positions of the elements appearing in the diagram, solid and broken arrows represent maps on  $E_0$  and  $E_1$ , respectively, and the curved lines show that, for example,  $j_0x = \delta''c + \delta's$ ; finally, the planes are, in order of increasing depth,  ${}_1E_0$ ,  ${}_2E_0$  and  ${}_3E_0$ .

*Proof.* Let  $\alpha \in {}_3E_2^{p, q}$  and  $a \in {}_3E_0^{p, q}$  be such that  $a \in \alpha$ ; then  $\delta'a = 0$  and there exists  $b \in {}_3E_0^{p-1, q+1}$  such that  $\delta'b = \delta''a$ . We see that  $\delta''b \in d_2\alpha$ . As  $k_1$  is an epimorphism, there exists  $c \in {}_2E_0^{p, q}$  and  $t \in {}_3E_0^{p-1, q}$  such that  $\delta'c = 0$  and  $k_0c = a + \delta't$ . We have

$$k_0\delta''c = \delta''k_0c = \delta''a + \delta'\delta't = \delta'(b - \delta''t),$$

and since  $\ker k_1 = \text{im } j_1$ , there are  $x \in {}_1E_0^{p, q+1}$  and  $s \in {}_2E_0^{p-1, q+1}$  with  $\delta'x = 0$  and  $j_0x = \delta''c + \delta's$ . Note that  $x \in \partial\alpha$ .

Observe now that  $j_0\delta''x = \delta''j_0x = \delta''\delta's = -\delta'\delta''s$ ; as  $j_1$  is a monomorphism, there is  $r \in {}_1E_0^{p-1, q+2}$  such that  $\delta'r = \delta''x$ . Then  $\delta''r \in d_2\partial\alpha$ .

Let now  $z = j_0r + \delta''s \in {}_2E_0^{p-1, q+2}$ . We have that

$$\delta'k_0s = k_0\delta's = k_0j_0x - k_0\delta''c = -\delta''k_0c = -\delta''a - \delta'\delta't = -\delta''a + \delta'\delta''t,$$

so  $\delta'(\delta''t - k_0s) = \delta''a$ . This implies that

$$k_0z = k_0\delta''s = \delta''k_0s = \delta''(k_0s - \delta''t) \in -d_2\alpha$$

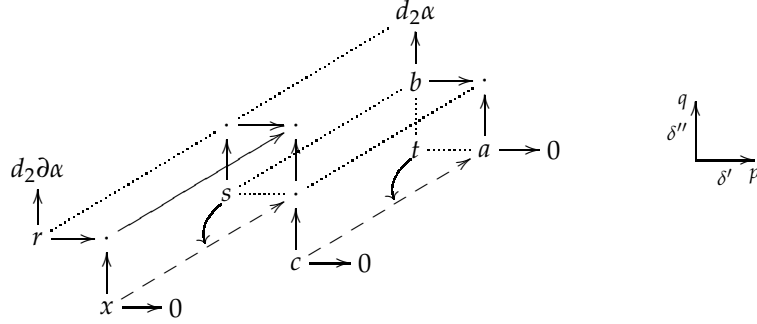


FIGURE 1. The chase in the proof of 2.2.1.

and because  $j_0\delta''r = \delta''j_0r = \delta''z$ ,  $\delta''r \in -\partial d_2\alpha$ . We thus see that  $d_2\partial\alpha = -\partial d_2\alpha$ , as we were required to show.  $\square$

**2.2.2.** We will apply the lemma in the context of the spectral sequence from 2.1.4. Consider a morphism of algebras  $\phi : A \rightarrow B$ , a left  $A$ -module  $M$  and a short exact sequence of left  $B$ -modules

$$0 \longrightarrow {}_1N \xrightarrow{j} {}_2N \xrightarrow{k} {}_3N \longrightarrow 0 \quad (3)$$

Let  $X^\bullet \twoheadrightarrow M$  be a projective resolution of  $M$ , and let us choose injective resolutions  ${}_iN \hookrightarrow {}_iY^\bullet$ , for  $1 \leq i \leq 3$ , and morphisms  $j^\bullet$  and  $k^\bullet$  among these such that in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_1N & \xrightarrow{j} & {}_2N & \xrightarrow{k} & {}_3N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}_1Y^\bullet & \xrightarrow{j^\bullet} & {}_2Y^\bullet & \xrightarrow{k^\bullet} & {}_3Y^\bullet \longrightarrow 0 \end{array}$$

the rows are exact and each square commutes. If we set

$${}_iZ^{\bullet,\bullet} = \text{hom}_B(B \otimes_A X^\bullet, {}_iY^\bullet)$$

for each  $i$  with  $1 \leq i \leq 3$ , we have, as one can easily show, an exact sequence of double complexes

$$0 \longrightarrow {}_1Z^{\bullet,\bullet} \longrightarrow {}_2Z^{\bullet,\bullet} \longrightarrow {}_3Z^{\bullet,\bullet} \longrightarrow 0$$

in which the morphisms are induced by  $j^\bullet$  and  $k^\bullet$ . We consider spectral sequences as in proposition 2.1.4.

For each bidegree, the exact sequence

$$0 \longrightarrow {}_1E_0^{p,q} \longrightarrow {}_2E_0^{p,q} \longrightarrow {}_3E_0^{p,q} \longrightarrow 0$$

is obtained by applying the functors  $\text{hom}_B(B \otimes_A X^q, -)$  to

$$0 \longrightarrow {}_1Y^p \longrightarrow {}_2Y^p \longrightarrow {}_3Y^p \longrightarrow 0$$

Since this last sequence splits, taking homology we get an exact sequence

$$0 \longrightarrow {}_1E_1^{p,q} \longrightarrow {}_2E_1^{p,q} \longrightarrow {}_3E_1^{p,q} \longrightarrow 0$$



We are then in the situation of the lemma and we see that the following diagram is anti-commutative:

$$\begin{array}{ccc}
 {}_3E_2^{p,q} \cong \text{Ext}_B^p(\text{Tor}_q^A(B, M), {}_3N) & \xrightarrow{\partial} & {}_1E_2^{p+1,q} \cong \text{Ext}_B^{p+1}(\text{Tor}_q^A(B, M), {}_1N) \\
 \downarrow d_2^{p,q} & & \downarrow d_2^{p+1,q} \\
 {}_3E_2^{p+2,q-1} \cong \text{Ext}_B^{p+2}(\text{Tor}_{q-1}^A(B, M), {}_3N) & \xrightarrow{\partial} & {}_1E_2^{p+3,q-1} \cong \text{Ext}_B^{p+3}(\text{Tor}_{q-1}^A(B, M), {}_1N)
 \end{array}$$

where  $\partial : {}_3E_2^{p,q} \rightarrow {}_1E_2^{p+1,q}$  is the connecting homomorphism for the long exact sequence corresponding to the  $\partial$ -functor  $\text{Ext}_B^\bullet(\text{Tor}_q^A(B, M), -)$  and the short exact sequence (3).

We are thus led to the following theorem:

**2.2.3. Theorem.** *Let  $\phi : A \rightarrow B$  be a morphism of algebras and let  $M$  be a left  $A$ -module. For each  $q \geq 0$  there is a class  $\mathcal{O}_q(M) \in \text{Ext}_B^2(\text{Tor}_{q-1}^A(B, M), \text{Tor}_q^A(B, M))$  such that, for each left  $B$ -module  $N$ , the  $E_2$  term of the spectral sequence of proposition 2.1.4 has differentials  $d_2^{p,q} : \text{Ext}_B^p(\text{Tor}_q^A(B, M), N) \rightarrow \text{Ext}_B^{p+2}(\text{Tor}_{q-1}^A(B, M), N)$  given by  $d_2^{p,q}(\zeta) = (-1)^p \zeta \circ \mathcal{O}_q(M)$ .*

*Proof.* The observations of 2.2.2, together with the naturality of the spectral sequences involved, imply that if we define  $\mathcal{O}_q(M)_N^p = (-1)^p d_2^{p,q}$  for each left  $B$ -module  $N$ , we obtain an operation  $\mathcal{O}_q(M) \in \text{sOp}_B^2(\text{Tor}_{q-1}^A(B, M), \text{Tor}_q^A(B, M))$ . The theorem follows now from the description given in 1.6 of this set.  $\square$

### 2.3. Hochschild cohomology.

**2.3.1.** Recall that the *Hochschild cohomology* of a  $k$ -algebra  $A$  is the functor of  $A$ -bimodules  $H^\bullet(A, -) = \text{Ext}_{A^e}^\bullet(A, -)$ ; here, as usual,  $A^e = A \otimes A^{\text{op}}$  is the so-called *enveloping algebra* of  $A$ , which is such that there is an isomorphism of categories  ${}_{A^e}\text{Mod} \cong {}_A\text{Mod}_A$ .

We write  $HH^\bullet(A) = H^\bullet(A, A)$  the value of Hochschild cohomology on the regular  $A$ -bimodule  $A \in {}_A\text{Mod}_A$ .

**2.3.2.** In this context, theorem 1.6 amounts to the following: given  $d \geq 0$ , there is a bijection between the class of stable operations  $H^\bullet(A, -) \rightarrow H^{\bullet+d}(A, -)$  and  $HH^d(A)$  and these bijections can be collected into an algebra isomorphism between the graded ring  $\text{sOp}_{A^e}^\bullet(A, A)$  of stable operations on  $H^\bullet(A, -)$  and the Yoneda algebra  $HH^\bullet(A)$ .

We remark that the Yoneda product on  $HH^\bullet(A)$  coincides with the classical cup product on Hochschild cohomology described in [12] and that the action of  $HH^\bullet(A)$  on  $H^\bullet(A, -)$  that corresponds under the isomorphism  $HH^\bullet(A) \cong \text{sOp}_{A^e}^\bullet(A, A)$  to the action of  $\text{sOp}_{A^e}^\bullet(A, A)$  on  $H^\bullet(A, -)$  can itself be computed using cup products.

**2.3.3.** We want to make explicit the specialization of 2.1.4 and 2.2.3 to Hochschild cohomology:

**Theorem.** *Let  $\phi : A \rightarrow B$  be a morphism of  $k$ -algebras. If  $M \in {}_B\text{Mod}_B$ , there is a convergent spectral sequence such that*

$$E_2^{p,q} \cong \text{Ext}_{B^e}^p(\text{Tor}_q^A(B, B), M) \Rightarrow H^\bullet(A, M)$$

*and this spectral sequence is functorial on  $M$ . Moreover, for each  $q \geq 0$  there exists  $\mathcal{O}_q(\phi) \in \text{Ext}_{B^e}^2(\text{Tor}_{q-1}^A(B, B), \text{Tor}_q^A(B, B))$  such that the differential of the term  $E_2$ ,*

$$d_2^{p,q} : \text{Ext}_{B^e}^p(\text{Tor}_q^A(B, B), M) \rightarrow \text{Ext}_{B^e}^{p+2}(\text{Tor}_{q-1}^A(B, B), M),$$

is given by

$$d_2^{p,q}(\zeta) = (-1)^p \zeta \circ \mathcal{O}_q(\phi)$$

if  $\zeta \in \text{Ext}_{B^e}^p(\text{Tor}_q^A(B, B), M)$ .

*Proof.* The morphism  $\phi$  induces in an obvious way a morphism  $\phi^e : A^e \rightarrow B^e$ , to which one can apply 2.1.4. This provides a spectral sequence with differentials of the form described in 2.2.3. The limit of the spectral sequence is  $H^\bullet(A, M)$ . On the other hand,  $\text{Tor}_\bullet^{A^e}(B^e, A) \cong \text{Tor}_\bullet^A(B, B)$  canonically, cf. [5, Corol. IX.4.4] and this shows that the  $E_2$  is of the form stated.  $\square$

**2.3.4.** This spectral sequence has good multiplicative properties. A particularly useful one is the following:

**Proposition.** *Let  $\phi : A \rightarrow B$  be a morphism of  $k$ -algebras such that  $B \otimes_A B \cong B$ . The edge morphism from the fiber to the limit in the spectral sequence described in 2.3.3 when  $M$  is the regular  $B$ -bimodule  $B$  is then a map*

$$e' : HH^\bullet(B) \rightarrow H^\bullet(A, B).$$

*This map is a morphism of algebras, when both its domain and its codomain are endowed with the cup product.*

*Proof.* This follows immediately from the description of  $e'$  given in 2.1.7 if one uses the standard Hochschild resolution in order to compute Ext-groups.  $\square$

**2.3.5.** When the morphism  $\phi$  is surjective, the class  $\mathcal{O}_1(\phi)$  is connected with a well-known construction:

**Proposition.** *Assume  $\phi : A \rightarrow B$  is a surjection of algebras with kernel  $I = \ker \phi$ . Then  $I/I^2$  is a  $B$ -bimodule in a natural way and there are isomorphisms  $\text{Tor}_0^A(B, B) \cong B$  and  $\text{Tor}_1^A(B, B) \cong I/I^2$ . Furthermore, the class*

$$\mathcal{O}_1(\phi) \in \text{Ext}_{B^e}^2(\text{Tor}_0^A(B, B), \text{Tor}_1^A(B, B)) = H^2(B, I/I^2)$$

*constructed in 2.3.3 coincides with the class of the singular extension of algebras*

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0 \quad (4)$$

*Proof.* The existence of the claimed isomorphisms is left as an easy exercise. The rest of the proposition can be proved by noting that the exact sequence for terms of lower degree in the spectral sequence  $E_2^{p,q} \cong \text{Ext}_{B^e}^p(\text{Tor}_q^A(B, B), -) \Rightarrow H^\bullet(A, -)$  constructed in the theorem is the analog for Hochschild cohomology of the 5-term exact sequence constructed for group cohomology in [19, Section VI.8] and then adapting the arguments given in [19, Section VI.10].  $\square$

**2.3.6.** We can construct explicitly a 2-extension of  $B$ -bimodules representing the class  $\mathcal{O}_1(\phi)$  appearing in 2.3.5. The construction is surely well-known but it does not appear in the standard references.

Recall from [23] that if  $\Lambda$  is an algebra, the  $\Lambda$ -bimodule of non-commutative differential forms  $\Omega(\Lambda)$  is the kernel of the multiplication map  $\mu : \Lambda \otimes \Lambda \rightarrow \Lambda$ . In particular, there is an exact sequence of  $\Lambda$ -bimodules

$$0 \longrightarrow \Omega(\Lambda) \longrightarrow \Lambda \otimes \Lambda \xrightarrow{\mu} \Lambda \longrightarrow 0 \quad (5)$$

There is a map  $d : \Lambda \rightarrow \Omega(\Lambda)$  given by  $d(\lambda) = \lambda \otimes 1 - 1 \otimes \lambda$ . One easily checks that this is a derivation and, moreover, it turns out that  $d$  is the universal derivation of  $\Lambda$  into  $\Lambda$ -bimodules, in the sense that composition with  $d$  induces an isomorphism  $\text{hom}_{\Lambda^e}(\Omega(\Lambda), -) \cong \text{Der}(\Lambda, -)$  of functors defined on  ${}_\Lambda \text{Mod}_\Lambda$ .

Let us put ourselves back in the situation of 2.3.5. Applying the functor  $B^e \otimes_{A^e}(-)$  to the short exact sequence (5) corresponding to  $\Lambda = A$  and using

the canonical isomorphism  $\mathrm{Tor}_1^{A^e}(B^e, A) \cong I/I^2$  noted above, we obtain an exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow B \otimes_A \Omega(A) \otimes_A B \longrightarrow B \otimes B \longrightarrow B \longrightarrow 0 \quad (6)$$

where the second map is induced by  $x \in I \mapsto 1 \otimes d(x) \otimes 1 \in B \otimes_A \Omega(A) \otimes_A B$  and the following two are induced from the corresponding maps in (5).

**Proposition.** *The class  $\mathcal{O}_1(\phi) \in H^2(B, I/I^2)$  constructed in 2.3.5 is represented by the 2-extension of  $B$ -bimodules (6).*

*Proof.* Let  $\sigma : B \rightarrow A/I^2$  be any  $k$ -linear section of the algebra map  $\bar{\phi} : A/I^2 \rightarrow B$  induced by  $\phi$ . Define now

$$c : b \otimes b' \in B \otimes B \mapsto \sigma(bb') - \sigma(b)\sigma(b') \in A/I^2.$$

Clearly  $\bar{\phi} \circ c = 0$ , so there exists a unique  $k$ -linear map  $\alpha : B \otimes B \rightarrow I/I^2$  such that  $c$  is the composition of  $\alpha$  and the inclusion  $I/I^2 \hookrightarrow A/I^2$ . A computation shows that  $\alpha$  is a 2-cocycle in the standard Hochschild complex which computes  $H^\bullet(B, I/I^2)$  and, indeed, its class in  $H^2(B, I/I^2)$  is the class corresponding to the extension (4).

Let us pick any  $k$ -linear section  $\iota : A/I^2 \rightarrow A$  for the projection  $A \rightarrow A/I^2$  and define maps

$$\hat{\alpha} : b \otimes b' \otimes b'' \otimes b''' \in B \otimes B \otimes B \otimes B \mapsto b\alpha(b' \otimes b'')b''' \in I/I^2$$

and

$$\hat{\sigma} : b \otimes b' \otimes b'' \in B \otimes B \otimes B \mapsto b \otimes d(\iota(\sigma(b'))) \otimes b'' \in B \otimes_A \Omega(A) \otimes_A B;$$

that this is well defined follows from a simple computation. Then we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & B^{\otimes 4} & \longrightarrow & B^{\otimes 3} & \longrightarrow & B \otimes B & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow \hat{\alpha} & & \downarrow \hat{\sigma} & & \parallel & & \parallel & & \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & B \otimes_A \Omega(A) \otimes_A B & \longrightarrow & B \otimes B & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

in which the top row is the standard Hochschild resolution of  $B$ . Recalling the way in which one shows that Yoneda functors are computable using projective resolutions, we see at once that this means that the 2-extension (6) represents the 2-cocycle  $\alpha$  and, hence, the class  $\mathcal{O}_1(\phi)$ .  $\square$

**2.3.7.** The classes  $\mathcal{O}_q(\phi) \in \mathrm{Ext}_{B^e}^2(\mathrm{Tor}_{q-1}^A(B, B), \mathrm{Tor}_q^A(B, B))$  constructed above for a surjective algebra map  $\phi : A \rightarrow B$  can be seen as higher order invariants for the extension associated to  $\phi$ , extending—according to 2.3.5—the classical invariant  $\mathcal{O}_1(\phi)$  constructed by Hochschild in [20].

**2.3.8.** Finally, we make explicit the 5-term exact sequence to which we made reference in the proof of 2.3.5 and describe the edge morphism which appears in it. It is obtained as usual from the spectral sequence constructed in 2.3.3 by looking at terms of low degree.

**Proposition.** *Let  $\phi : A \rightarrow B$  be a surjection of algebras and let  $M$  be a  $B$ -bimodule. There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^1(B, M) \longrightarrow H^1(A, M) \xrightarrow{e} \mathrm{hom}_{B^e}(I/I^2, M) \longrightarrow \\ \longrightarrow H^2(B, M) \longrightarrow H^2(A, M) \end{aligned}$$

The edge morphism  $e$  is induced by the map  $\text{Der}(A, M) \rightarrow \text{hom}_{B^e}(I/I^2, M)$  which sends  $f \in \text{Der}(A, M)$  to the composition

$$I/I^2 \longrightarrow B \otimes_A \Omega(A) \otimes_A B \xrightarrow{1 \otimes f \otimes 1} B \otimes_A M \otimes_A B \xrightarrow{\cong} M$$

The first map here is the one appearing in (6).  $\square$

### 3. THE COHOMOLOGY OF MONOGENIC ALGEBRAS

**3.1.** We fix a monic polynomial  $f = \sum_{i=0}^N \alpha_i X^i \in k[X]$  of degree  $N$  and consider the algebra  $A = k[X]/(f)$ ; let  $x$  be the class of  $X$  in  $A$ . We want to describe its Hochschild cohomology with values in an  $A$ -bimodule  $M$ . To do so, we consider the canonical projection  $\phi : k[X] \rightarrow A$  and study the spectral sequence  $E$  attached to it in 2.3.3.

**3.2.** The initial term  $E_2$ , which has  $E_2^{p,q} = \text{Ext}_{A^e}^p(\text{Tor}_q^{k[X]}(A, A), M)$ , is easy to compute. Indeed, the obvious short exact sequence

$$0 \longrightarrow k[X] \xrightarrow{f} k[X] \longrightarrow A \longrightarrow 0 \quad (7)$$

provides a projective resolution of  $A$  as a  $k[X]$ -module either on the left or on the right; using it to compute  $\text{Tor}_\bullet^{k[X]}(A, A)$ , we immediately see that

$$\text{Tor}_p^{k[X]}(A, A) \cong \begin{cases} A, & \text{if } p = 0 \text{ or } p = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This implies that there are isomorphisms

$$E_2^{p,q} \cong \begin{cases} H^p(A, M), & \text{if } q = 0 \text{ or } q = 1; \\ 0, & \text{otherwise.} \end{cases}$$

**3.3.** We look now at the limit of  $E$ , which is  $H^\bullet(k[X], M)$ . There is a projective resolution

$$0 \longrightarrow k[X] \otimes k[X] \xrightarrow{d_1} k[X] \otimes k[X] \xrightarrow{d_0} k[X]$$

of  $k[X]$  as a  $k[X]$ -bimodule, with  $d_1(1 \otimes 1) = X \otimes 1 - 1 \otimes X$  y  $d_0(1 \otimes 1) = 1$ . A trivial computation using it shows that

$$H^p(k[X], M) \cong \begin{cases} M^x, & \text{if } p = 0; \\ M_x, & \text{if } p = 1; \\ 0, & \text{if } p \geq 2. \end{cases}$$

Here we are writing  $M^x \doteq \{m \in M : xm = mx\}$  and  $M_x \doteq \frac{M}{\{xm - mx : m \in M\}}$ .

**3.4.** We conclude that the  $E_2$  term looks like this:

We have written the limit of this sequence,  $H^\bullet(k[X], M)$ , under the  $p$ -axis.

Convergence then implies that  $H^0(A, M) \cong M^x$ , that we have an exact sequence

$$0 \longrightarrow H^1(A, M) \longrightarrow H^1(k[X], M) \xrightarrow{e} H^0(A, M) \xrightarrow{d_2^{0,1}} H^2(A, M) \longrightarrow 0$$

with  $e$  the edge morphism described in 2.1.7 and that the differentials provide isomorphisms  $d_2^{p,1} : H^p(A, M) \xrightarrow{\cong} H^{p+2}(A, M)$  for each  $p \geq 1$ .

According to 2.3.3,  $d_2^{p,1}$  is given by Yoneda multiplication by a class

$$\zeta \in \text{Ext}_{A^e}^2(\text{Tor}_0^{k[X]}(A, A), \text{Tor}_1^{k[X]}(A, A)) = HH^2(A)$$

and, in fact, looking back at the proof of 1.6, we see that  $\zeta = d_2^{0,1}(1_A)$  with  $1_A \in HH^0(A) = A$ .

On the other hand, it is easy to follow the recipe given in 2.1.7 in order to see that the edge morphism  $e$  is induced on  $H^1(k[X], M) = M^x$  by the map  $\hat{e} : M \rightarrow M_x$  given by

$$\hat{e}(m) = \sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} \alpha_i x^s m x^t.$$

**3.5.** In particular, if  $M = A$  is the regular  $A$ -bimodule, of course we have that  $A_x = A^x = A$ ,  $e : A \rightarrow A$  is just multiplication by  $f'$ , and we see that  $HH^0(A) \cong A$ ,  $HH^1(A) \cong A_{f'} \doteq \ker e$ ,  $HH^2(A) \cong A/(f')$  and that multiplication by  $\zeta = d_2^{0,1}(1) \in HH^2(A)$  gives an epimorphism  $HH^0(A) \rightarrow HH^2(A)$  and, for each  $p \geq 1$ , an isomorphism  $HH^p(A) \rightarrow HH^{p+2}(A)$ .

**3.6.** Under our isomorphisms,  $\zeta$  corresponds to the class of 1 in  $HH^2(A) \cong A/(f')$ . If we put  $d = \gcd(f, f')$  and let  $q$  be such that  $f = qd$ , there is an isomorphism  $A_{f'} \cong (q)/(f)$ . In particular  $HH^1(A)$  is a cyclic  $A$ -module generated by an element  $\tau$ , corresponding to the class of  $q$  under this isomorphism, with annihilator  $(d) \triangleleft A$ .

**3.7.** It is quite clear that the action of  $HH^0(A) = A$  on  $HH^\bullet(A)$  corresponds, under our isomorphisms, to the obvious structure of  $A$ -modules on  $A_{f'}$  and  $A/(f')$ . Since we understand multiplication by  $\zeta$ , to describe the multiplicative structure on  $HH^\bullet(A)$ , we need only concentrate on computing  $\tau^2$ .

**3.8.** Assume for a moment that 2 is invertible in  $k$ . The graded commutativity of  $HH^\bullet(A)$  and the fact that  $|\tau| = 1$  immediately imply, then, that  $\tau^2 = 0$  and we see that we have in this situation an isomorphism  $HH^\bullet(A) \cong k[x, \tau, \zeta]/(f, \tau d, \zeta f')$  of graded commutative algebras.

**3.9.** In the general case, we have the following theorem:

**Theorem.** Let  $f = \sum_{i=0}^N \alpha_i X^i \in k[X]$  be a monic polynomial of degree  $N$  and consider the  $k$ -algebra  $A = k[X]/(f)$ . Let  $d = \gcd(f, f')$ , let  $q \in k[X]$  be such that  $f = qd$  and put

$$u = q^2 \sum_{i=0}^N \alpha_i \frac{i(i-1)}{2} X^{i-2}.$$

Then there is an isomorphism of graded commutative algebras

$$HH^\bullet(A) \cong k[x, \tau, \zeta]/(f(x), \tau d(x), \zeta f'(x), \tau^2 - u(x)\zeta),$$

where the generators in the right hand side have degrees  $|x| = 0$ ,  $|\tau| = 1$  and  $|\zeta| = 2$ .

*Proof.* At this point, we need only show that  $\tau^2 = u(x)\zeta$ . We will resort to a direct computation: we have not been able to find a more conceptual argument in the spirit of those used above to handle this.

There is a commutative diagram of  $A$ -bimodule morphisms

$$\begin{array}{ccccccc}
 A \otimes A & \xrightarrow{d_3} & A \otimes A & \xrightarrow{d_2} & A \otimes A & \xrightarrow{d_1} & A \otimes A \xrightarrow{m} A \\
 \downarrow s_3 & & \uparrow r_2 \downarrow s_2 & & \uparrow r_1 \downarrow s_1 & & \uparrow r_0 \downarrow s_0 \\
 A \otimes A^{\otimes 3} \otimes A & \xrightarrow{d'_3} & A \otimes A^{\otimes 2} \otimes A & \xrightarrow{d'_2} & A \otimes A \otimes A & \xrightarrow{d'_1} & A \otimes A \xrightarrow{m} A
 \end{array} \quad (8)$$

where  $m : A \otimes A \rightarrow A$  is the multiplication map, the maps  $d'_p$  are the Hochschild boundary maps,  $s_0 = r_0 = \text{id}$  and

$$d_p(1 \otimes 1) = 1x \otimes 1 - 1 \otimes x1 \quad \text{if } p \text{ is odd,}$$

$$d_p(1 \otimes 1) = \sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} \alpha_i 1x^s \otimes x^t1 \quad \text{if } p \text{ is even,}$$

$$s_1(1 \otimes 1) = 1 \otimes x \otimes 1,$$

$$s_2(1 \otimes 1) = \sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} \alpha_i 1 \otimes x^s \otimes x \otimes x^t,$$

$$s_3(1 \otimes 1) = \sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} \alpha_i 1 \otimes x \otimes x^s \otimes x \otimes x^t,$$

$$r_1(1 \otimes x^i \otimes 1) = \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} x^s \otimes x^t,$$

$$r_2(1 \otimes x^i \otimes x^j \otimes 1) = -1 \otimes q_f(X^i + X^j).$$

Here we are using functions  $q_f, r_f : k[X] \rightarrow k[X]$  defined so that, for all  $h \in k[X]$ ,  $h = q_f(h)f + r_f(h)$  and either  $r_f(h) = 0$  or  $\deg r_f(h) < \deg f$ .

The rows in (8) are exact and, in fact, they are the beginnings of two projective resolutions of  $A$  as an  $A$ -bimodule: the lower row comes from the usual Hochschild resolution  $A^{\otimes(\bullet+2)}$  of  $A$  and the upper row comes from the well-known 2-periodic resolution  $P_\bullet$  of  $A$  constructed in [4]. The vertical maps are the first components of a comparison of resolutions. The complete picture can be found in [22], but we will not make use of it.

It is clear that  $\tau \in HH^1(A)$  is the class of the unique derivation  $t : A \rightarrow A$  such that  $t(x) = q$ . This implies that  $\tau$  can be seen as the class of the 1-cocycle  $t \in \text{hom}(A, A)$  of the complex  $\text{hom}(A^{\otimes \bullet}, A) \cong \text{hom}_{A^e}(A^{\otimes(\bullet+2)}, A)$  constructed by applying  $\text{hom}_{A^e}(-, A)$  to the Hochschild resolution of  $A$ .

Now, when one sees  $HH^\bullet(A)$  as the cohomology of this complex, products can be computed using the cup product  $\smile$  introduced in [12]. This means that  $\tau^2$  is the class of  $t \smile t : A \otimes A \rightarrow A$  in  $HH^2(A)$ , where

$$(t \smile t)(a \otimes b) = t(a)t(b) = a'b'q^2,$$

for all  $a, b \in A$ ; here derivatives are taken on arbitrary representatives for elements of  $A$  in  $k[X]$ .

Pulling back the 2-cocycle  $t \smile t$  from  $\text{hom}_{A^e}(A^{\otimes(\bullet+2)}, A)$  to the top row in (8) along the given comparison morphisms immediately shows that  $\tau^2$  is represented

by the unique 2-cocycle of the complex  $\text{hom}_{A^e}(P_\bullet, A)$  which maps  $1 \otimes 1 \in P_2 = A \otimes A$  to

$$\sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ u+v+1=i}} \alpha_i t(x^u)t(x)x^v = \sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ u+v+1=i}} \alpha_i u x^{i-2} q^2 = \sum_{i=0}^N \alpha_i \frac{i(i-1)}{2} x^{i-2} q^2.$$

Since the class  $\zeta$  is represented in the complex  $\text{hom}_{A^e}(P_\bullet, A)$  by the 2-cocycle  $z : A \otimes A \rightarrow A$  such that  $z(1 \otimes 1) = 1$ , we see that  $t \smile t = u(x)z$  and, then, that  $\tau^2 = u(x)\zeta$ , as stated in 3.9. This completes the proof of that theorem.  $\square$

**3.10.** If 2 is invertible in  $k$ , then the polynomial  $u$  appearing in 3.9 is simply  $\frac{1}{2}q^2 f''$  and it is an easy exercise to show that this is zero in  $A/(f')$ . This corresponds, of course, to the observation made in 3.8.

**3.11.** Before passing on to other matters, we take the opportunity of computing the rest of the “cohomology structure” of our algebra  $A$  in the sense used in [12]. We will use the conventions and notations of [13, Section 4], which are, by now, rather standard; in particular, we use the composition products  $\circ$  and  $\circ_i$  for cocycles on the Hochschild resolution.

**Theorem.** *In the situation of theorem 3.9, let*

$$w = \sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} \alpha_i q_f((s+1)x^s q)x^t.$$

*Then the Gerstenhaber bracket on  $HH^\bullet(A)$  is completely determined by the relations*

$$\begin{aligned} [\tau, x] &= q, \\ [\zeta, \tau] &= w\zeta, \\ [x, x] &= [\tau, \tau] = [\tau, \zeta] = [x, \zeta] = 0. \end{aligned}$$

*Proof.* The Gerstenhaber bracket is graded on  $HH^\bullet(A)[1]$ , so that it is clear that  $[x, x] \in HH^{-1}(A)$  and  $[\tau, \tau]$  are zero. The other four relations will be established by computation. We use the maps  $r_\bullet$  and  $s_\bullet$  to go from cocycles on the top row of (8) to cocycles on the bottom row and back, respectively, and identify  $\text{hom}_{A^e}(A^{\otimes(\bullet+2)}, A)$  with  $\text{hom}(A^{\otimes\bullet}, A)$  as usual. Also, we will write

$$\int \phi(i, s, t) \doteq \sum_{i=0}^N \sum_{\substack{s,t \geq 0 \\ s+t+1=i}} \phi(i, s, t)$$

for functions  $\phi$  defined on non-negative integers.

We have that

$$\begin{aligned} s_0^*([r_0^*(x), r_1^*(\tau)])(1 \otimes 1) &= [r_0^*(x), r_1^*(\tau)](1) \\ &= (r_1^*(\tau) \circ (r_0^*(x)))(1) \\ &= r_1^*(\tau)(r_0^*(x)(1)) \\ &= r_1^*(\tau)(x) \\ &= q, \end{aligned}$$

so  $[x, \tau] = q$ . Similarly, we see that

$$\begin{aligned} s_1^*([r_0^*(x), r_2^*(\zeta)])(1 \otimes 1) &= [r_0^*(x), r_2^*(\zeta)](x) \\ &= (r_2^*(\zeta) \circ r_0^*(x))(x) \\ &= r_2^*(\zeta)(r_0^*(x), x) - r_0^*(\zeta)(x, r_0^*(x)) \\ &= 0, \end{aligned}$$

because  $r_2^*(\zeta) : A^{\otimes 2} \rightarrow A$  is symmetric. This tells us that  $[x, \zeta] = 0$ .

Using symmetry again, we compute

$$\begin{aligned} s_3^*([r_2^*(\zeta), r_2^*(\zeta)])(1 \otimes 1) &= \int \alpha_i [r_2^*(\zeta), r_2^*(\zeta)](x \otimes x^s \otimes x) x^t \\ &= 2 \int \alpha_i (r_2^*(\zeta) \circ r_2^*(\zeta))(x \otimes x^s \otimes x) x^t \\ &= 2 \int \alpha_i \left( r_2^*(\zeta)(r_2^*(\zeta)(x \otimes x^s) \otimes x) - r_2^*(\zeta)(x \otimes r_2^*(\zeta)(x \otimes x^s)) \right) x^t \\ &= 0, \end{aligned}$$

so we see that  $[\zeta, \zeta] = 0$ .

Finally,

$$\begin{aligned} s_2^*([r_1^*(\tau), r_2^*(\zeta)])(1 \otimes 1) &= \int \alpha_i [r_1^*(\tau), r_2^*(\zeta)](x^s \otimes x) x^t \\ &= \int \alpha_i \left( q_f((s+1)x^s q) - q_f(x^{s+1})' q \right) x^t \end{aligned}$$

and, since  $q_f(x^{s+1})' = 0$  if  $0 \leq s < N$ , this is

$$= \int \alpha_i q_f((s+1)x^s q) x^t.$$

We have thus verified all the relations claimed in the theorem.  $\square$

#### 4. VARIATIONS

The computation done in section 3 was successful because of the many favorable traits of the situation under consideration. It turns out, though, that a similar line of reasoning can be applied in various other less favorable contexts in order to obtain useful information on cohomology. We collect here a few examples.

**4.1. Proposition.** *Let  $A$  be an algebra and  $I \triangleleft A$  an ideal which is flat as an  $A$ -module on the left or on the right and put  $B = A/I$ . Then  $H^0(B, -) \cong H^0(A, -)$  and there is a long exact sequence*

$$\begin{aligned} 0 \rightarrow H^1(B, M) \rightarrow H^1(A, M) \rightarrow \text{Ext}_{B^e}^0(I/I^2, M) \xrightarrow{\sim \zeta} H^2(B, M) \rightarrow \cdots \\ \cdots \rightarrow H^p(B, M) \rightarrow H^p(A, M) \rightarrow \text{Ext}_{B^e}^{p-1}(I/I^2, M) \xrightarrow{\sim \zeta} H^{p+1}(B, M) \rightarrow \cdots \end{aligned}$$

functorial on  $B$ -bimodules  $M$ . Here  $\zeta \in H^2(B, I/I^2)$  is the class of the singular extension

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0$$

In particular, restriction of scalars induces functorial isomorphisms

$$H^p(B, -) \rightarrow H^p(A, -)$$

on  $B$ -bimodules for  $p > \text{pdim}_{A^e} I/I^2 + 1$ .

*Proof.* Looking at the long exact sequence for  $\text{Tor}_\bullet^A(B, -)$  corresponding to

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

we see that  $\text{Tor}_p^A(B, B) \cong \text{Tor}_{p-1}^A(B, I) = 0$  for  $p > 1$ . Using this together with the convergence of the spectral sequence in 2.3.3 we see that the long exact sequence in the statement exists. All other claims follow at once.  $\square$



**4.2.** Let  $A$  be an algebra, let  $x \in A$  be *normal* (so that  $Ax = xA$ ) and assume moreover that  $x$  is not a divisor of zero. Normality implies that the left ideal  $I = Ax$  is actually a bilateral ideal and we can consider the quotient algebra  $B = A/I$ . The map  $r_x : A \rightarrow I$  given by right multiplication by  $x$  is an isomorphism of left  $A$ -modules—in particular,  $I$  is flat and we can apply 4.1 to this situation.

One can see at once that the hypothesis on  $x$  implies that there exists a unique automorphism  $\alpha \in \text{Aut}_{\text{Alg}}(A)$  such that  $ax = x\alpha(a)$  for all  $a \in A$ ; notice that  $\alpha = \text{id}_A$  iff  $x$  is central. Moreover,  $\alpha(I) = I$  so  $\alpha$  induces an automorphism of  $B$ , which we denote  $\alpha$  as well. If  $M$  is a  $B$ -bimodule, we write  $M_\alpha$  the  $B$ -bimodule which coincides with  $M$  as a left  $B$ -module and whose right action is that of  $M$  ‘twisted’ by  $\alpha$ , so that

$$m \cdot b = m\alpha(b), \quad \forall m \in M_\alpha, b \in B.$$

Clearly,  $r_x(I) = I^2$  and in fact  $r_x$  induces an isomorphism of  $B$ -bimodules  $I/I^2 \cong B_\alpha$ . Now, it is easy to see that for each  $B$ -bimodule  $M$ , there is a natural isomorphism  $\text{Ext}_{B^e}^\bullet(B_\alpha, M) \cong \text{Ext}_{B^e}^\bullet(B, M_{\alpha^{-1}}) = H^\bullet(B, M_{\alpha^{-1}})$ ; indeed, this follows from the fact that the functor  $(-) \otimes_B B_\alpha : {}_B\text{Mod}_B \rightarrow {}_B\text{Mod}_B$  is an equivalence which maps  $B$  to  $B_\alpha$  and  $M_{\alpha^{-1}}$  to  $M$ . Using this, the long exact sequence of 4.1 becomes

$$\begin{aligned} 0 \longrightarrow H^1(B, M) \longrightarrow H^1(A, M) \longrightarrow H^0(B, M_{\alpha^{-1}}) \xrightarrow{\sim \zeta} H^2(B, M) \longrightarrow \cdots \\ \cdots \longrightarrow H^p(B, M) \longrightarrow H^p(A, M) \longrightarrow H^{p-1}(B, M_{\alpha^{-1}}) \xrightarrow{\sim \zeta} H^{p+1}(B, M) \longrightarrow \cdots \end{aligned} \quad (9)$$

The following proposition records an interesting special case which occurs when  $\text{pdim}_{A^e} A \leq 1$ —for example, when  $A$  is finite dimensional and hereditary.

**4.3. Proposition.** *Let  $A$  be an algebra such that  $\text{pdim}_{A^e} A \leq 1$ . Let  $x \in A$  be a normal non-zero divisor and let  $\alpha \in \text{Aut}_{\text{Alg}}(A)$  be such that  $ax = x\alpha(a)$  for all  $a \in A$ . Put  $I = (x)$  and  $B = A/I$ . Then for each  $B$ -bimodule  $M$  and each  $p \geq 0$  there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^{2p+1}(B, M) \longrightarrow H^1(A, M_{\alpha^{-p}}) \xrightarrow{e} \\ \longrightarrow H^0(A, M_{\alpha^{-p-1}}) \longrightarrow H^{2p+2}(B, M) \longrightarrow 0 \end{aligned}$$

We thus see that under the stated conditions, computation of cohomology is reduced to the consideration of what happens in low degrees.

*Proof.* As  $\text{pdim}_{A^e} A \leq 1$ , the long exact sequence (9) collapses into an exact sequence

$$0 \longrightarrow H^1(B, M) \longrightarrow H^1(A, M) \longrightarrow H^0(B, M_{\alpha^{-1}}) \longrightarrow H^2(B, M) \longrightarrow 0$$

and natural isomorphisms  $H^p(B, M_{\alpha^{-1}}) \cong H^{p+2}(B, M)$  for all  $p \geq 1$ . Iterating these isomorphisms, using the exact sequence and taking into account the isomorphism  $H^0(A, -) \cong H^0(B, -)$ , we obtain the sequences referred to in the proposition.  $\square$

**4.4.** The proof of the proposition also shows the following:

**Corollary.** *With the notations of 4.3, assume that  $\alpha^n = \text{id}_A$ . Then there is a class  $\zeta \in HH^{2n}(B)$  such that  $\smile \zeta : H^p(B, M) \rightarrow H^{p+2n}(B, M)$  is an isomorphism for all  $p \geq 1$  and all  $B$ -bimodules  $M$ . In particular, if  $\dim_k B < \infty$ , then  $HH^\bullet(B)$  is a finitely generated algebra and  $H^\bullet(B, M)$  is a finitely generated  $HH^*(B)$ -module for all finite dimensional  $B$ -bimodules  $M$ .*

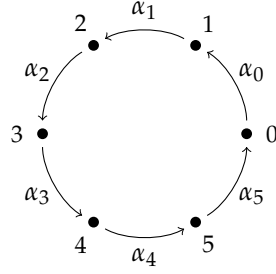


FIGURE 2. The 6-crown.

*Proof.* The class  $\zeta \in H^2(B, I/I^2)$  of the extension

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0$$

is such that  $\smile \zeta : H^p(B, M) \rightarrow H^{p+2}(B, M_\alpha)$  is an isomorphism for all  $B$ -bimodules  $M$  and all  $p \geq 1$ . Now the  $n$ -th Yoneda power of  $\zeta$  can be seen as a class  $\zeta^n = \zeta^n \in H^2(B, (I/I^2)^{\otimes_{B^n}}) = H^2(B, B_{\alpha^n}) = HH^{2n}(B)$  and, of course, the map  $\smile \zeta^n : H^p(B, M) \rightarrow H^{p+2n}(B, M)$  is an isomorphism for all  $p \geq 0$ . This proves the first claim in the statement.

Assume now that  $\dim_k B < \infty$  and let  $M$  be a finite dimensional  $B$ -bimodule. What we have so far implies that  $H^\bullet(B, M)$  is generated as a  $HH^\bullet(B)$ -module by  $\bigoplus_{p=0}^{2n-1} H^p(B, M)$ , which is a finite dimensional vector space. It follows at once that  $H^\bullet(B, M)$  is a finitely generated module. Similarly,  $HH^\bullet(B)$  is generated as an algebra by  $\bigoplus_{p=0}^{2n-1} HH^p(B)$  together with  $\zeta$ , so it is itself finitely generated.  $\square$

**4.5.** In some cases, we can obtain more precise information about the multiplicative structure on Hochschild cohomology in the situation of 4.2. We consider here only a very simple instance.

Let  $n \in \mathbb{N}$  and let  $Q$  be the quiver with vertex set  $Q_0 = \mathbb{Z}_n$  and arrow set  $Q_1 = \{\alpha_i\}_{i \in \mathbb{Z}_n}$  such that  $\alpha_i$  starts at the vertex  $i$  and ends at the vertex  $i+1$ . This is sometimes called the  $n$ -crown quiver. See figure 2 for a drawing of  $Q$  when  $n = 6$ .

Let  $kQ$  be the path algebra on  $Q$  and put  $t = \sum_{\alpha \in Q_1} \alpha \in kQ$ . If  $f \in k[X]$  is a non-zero polynomial, one easily sees that  $u = f(t) \in kQ$  is a regular normal element and if we put  $I = (u)$  and  $B = kQ/I$ , the observations in 4.2 provide a long exact sequence relating  $HH^\bullet(B)$  and  $H^\bullet(kQ, B)$ .

We take  $f = X^l$  with  $l = nm - 1$  for some  $m \geq 1$ , so that  $B$  is one of the symmetric truncated cycle algebras considered in [3] or the special Brauer tree algebras studied in [21]. Additionally, we assume  $n \geq 2$ : the case in which  $n = 1$  was treated in the previous section.

The choice of  $l$  implies that  $\alpha = \text{id}_A$ . Using this and that  $\text{pdim}_{(kQ)^e} kQ = 1$ , we see that the long exact sequence (9) degenerates, as in the proof of 4.3, into, on one hand, an exact sequence

$$0 \longrightarrow HH^1(B) \longrightarrow H^1(kQ, B) \xrightarrow{e} HH^0(B) \xrightarrow{\smile \zeta} HH^2(B) \longrightarrow 0 \quad (10)$$

with  $e$  the edge morphism from the limit to the ‘base’ described in 2.1.7 and  $\zeta \in HH^2(B)$  the class of the extension of algebras

$$0 \longrightarrow B \cong I/I^2 \longrightarrow kQ/I^2 \longrightarrow B \longrightarrow 0 \quad (11)$$

and, on the other hand, isomorphisms  $HH^p(B) \rightarrow HH^{p+2}(B)$  for  $p \geq 1$ , all given by cup-product with  $\zeta$ .

**Proposition.** *There is an isomorphism of algebras  $HH^{\text{ev}}(B) \cong Z(B)[\zeta]$ , there is an isomorphism of  $HH^{\text{ev}}(B)$ -modules  $HH^{\text{odd}}(B) \cong H^1(kQ, B)[\zeta]$  and the product of two elements of  $HH^{\text{odd}}(B)$  is zero. This information, together with the usual  $Z(B)$ -bimodule structure on  $H^1(kQ, B)$ , completely determines the cohomology algebra*

$$HH^\bullet(B) \cong Z(B)[\zeta] \oplus H^1(kQ, B)[\zeta].$$

*Proof.* Let  $E \subset kQ$  be the subalgebra generated by the vertices. One easily checks that, because  $n \geq 2$ , every  $E$ -linear derivation  $kQ \rightarrow B$  into the  $kQ$ -bimodule  $B$  induces an  $E$ -linear derivation  $B \rightarrow B$  simply by composing with the projection  $\phi$ . This means that the map  $HH^1(B) \rightarrow H^1(kQ, B)$  in (10) is surjective and, of course, it follows that  $e = 0$  and that  $\smile \zeta : HH^0(B) \rightarrow HH^2(B)$  is an isomorphism. We can conclude that  $HH^{\text{ev}}(B) \cong Z(B)[\zeta]$ . We already know how  $\zeta$  acts on  $HH^{\text{odd}}(B)$  and the center  $Z(B) = HH^0(B)$  acts on it canonically, so we need only describe the restriction of the cup-product to  $HH^1(B) \otimes HH^1(B)$  in order to complete the proof.

Let us write  $\gamma_i^j$  the only path in  $Q$  of length  $j$  which starts in the vertex  $i \in \mathbb{Z}_n$ . Then  $\{\gamma_i^j : i \in \mathbb{Z}_n, j \geq 0\}$  is a basis of  $kQ$  and  $\{\gamma_i^j : i \in \mathbb{Z}_n, 0 \leq j < l\}$  is a basis of  $B$ . Let  $kQ_1$  and  $R$  be the sub- $E^e$ -modules of  $kQ$  spanned by  $Q_1$  and  $\{\gamma_i^l : i \in \mathbb{Z}_n\}$ , respectively. We consider the following commutative diagram

$$\begin{array}{ccccccc} B \otimes_E R \otimes_E B & \xrightarrow{d_2} & B \otimes_E kQ_1 \otimes_E B & \xrightarrow{d_1} & B \otimes_E B & \xrightarrow{\mu} & B \\ \downarrow s_2 & & \uparrow r_1 \downarrow s_1 & & \parallel & & \parallel \\ B \otimes_E^4 & \longrightarrow & B \otimes_E^3 & \longrightarrow & B \otimes_E B & \xrightarrow{\mu} & B \end{array}$$

in which the bottom row is the standard Hochschild resolution of  $B$  taken over  $E$ , the map  $\mu$  is the multiplication,  $s_1$  is the inclusion and

$$\begin{aligned} d_1(1 \otimes \alpha \otimes 1) &= \alpha \otimes 1 - 1 \otimes \alpha, \\ d_2(1 \otimes \gamma_i^l \otimes 1) &= \sum_{s=0}^{l-1} \gamma_{i+s+1}^{l-s-1} \otimes \alpha_{i+s} \otimes \gamma_i^s, \\ r_1(1 \otimes \gamma_i^j \otimes 1) &= \sum_{s+t+1=j} \gamma_{i+t+1}^s \otimes \alpha_{i+t} \otimes \gamma_i^t, \\ s_2(1 \otimes \gamma_i^l \otimes 1) &= \sum_{s+t+1=l} 1 \otimes \gamma_{i+t+1}^s \otimes \alpha_{i+t} \otimes \gamma_i^t, \end{aligned}$$

The top row is the beginning of the Bardzell resolution for  $B$ , cf. [2].

Let us take now classes  $\phi, \psi \in HH^1(B)$  and 1-cocycles  $f, g : B \otimes_E kQ_1 \otimes_E B \rightarrow B$  defined on the Bardzell complex such that  $f$  and  $g$  represent  $\phi$  and  $\psi$ , respectively. We identify  $f$  and  $g$  canonically to elements of  $\text{hom}_{E^e}(Q_1, B)$ , which can itself be seen as the vector space of  $E$ -linear derivations  $kQ \rightarrow B$ . Then  $s_2^*(r_1^*(f) \smile r_1^*(g))$  is a representative for  $\phi \smile \psi$ . Since

$$\begin{aligned} s_2^*(r_1^*(f) \smile r_1^*(g))(1 \otimes \gamma_i^l \otimes 1) \\ = \sum_{s+1+t+1+v=l} \gamma_{i+1+t+1+v}^s f(\alpha_{i+t+1+v}) \gamma_{i+1+v}^t g(\alpha_{i+v}) \gamma_i^v \end{aligned}$$

and every element of  $\text{hom}_{E^e}(Q_1, B)$  maps arrows to linear combinations of paths of length at least one (because  $n \geq 2$ ) we see that  $s_2^*(r_1^*(f) \smile r_1^*(g)) = 0$ , so in

particular,  $\phi \smile \psi = 0$  in  $HH^2(B)$ . This implies that the cup product vanishes on  $HH^{\text{odd}}(B)$ .  $\square$

## 5. THE COHOMOLOGY OF NICE QUOTIENTS

**5.1.** It is well known that a morphism of algebras  $\phi : A \rightarrow B$  is an epimorphism in the category  $\text{Alg}$  of algebras iff extension of scalars along  $\phi$  is a full and faithful functor  $\phi^* : {}_B\text{Mod} \rightarrow {}_A\text{Mod}$ ; in particular, a surjection is an epimorphism.

Following [11], one says that  $\phi$  is a *homological epimorphism* if the corresponding functor on bounded derived categories  $D^b(\phi^*) : D^b({}_B\text{Mod}) \rightarrow D^b({}_A\text{Mod})$  is full and faithful. An easy induction on the length of complexes shows that this is equivalent to the condition that extension of scalars induce an isomorphism  $\text{Ext}_B^\bullet(-, -) \rightarrow \text{Ext}_A^\bullet(-, -)$  of bifunctors of  $B$ -modules.

**5.2.** The kernel of a homological epimorphism is called a *homological ideal*.

Homological ideals were first considered by Maurice Auslander *et al.* in [1] under the name of *strong idempotent ideals*: indeed, it turns out that ideals of this form are idempotent, cf. 5.6.

**5.3.** The characterization of homological epimorphisms given in the following proposition comes from [11] and [26, Proposition 2.3], though the fact that  $(d) \Rightarrow (a)$  was not noted there. The final statement about Hochschild cohomology strengthens [26, Proposition 3.1]; in particular, remark that we are not restricting ourselves to finite dimensional algebras.

**Proposition.** *Let  $\phi : A \rightarrow B$  be an morphism of algebras such that  $B \otimes_A B \cong B$  as  $B$ -bimodules. The following statements are equivalent:*

- (a)  $\phi : A \rightarrow B$  is a homological epimorphism;
- (b)  $\text{Tor}_+^A(B, M) = 0$  for all  $M \in {}_B\text{Mod}$ ;
- (c)  $\text{Tor}_+^A(B, B) = 0$ ;
- (d)  $\phi^e : A^e \rightarrow B^e$  is a homological epimorphism.

When they are satisfied, there is a natural isomorphism

$$H^\bullet(B, -) \rightarrow H^\bullet(A, -) \tag{12}$$

of functors of  $B$ -bimodules. Furthermore, the specialization of (12) to the  $B$ -bimodule  $B$  is an isomorphism of algebras  $HH^\bullet(B) \rightarrow H^\bullet(A, B)$ .

*Proof.* We prove  $(a) \Rightarrow (b)$ . Fix  $M \in {}_B\text{Mod}$  and consider the functorial spectral sequence with  $E_2^{p,q}(-) \cong \text{Ext}_B^p(\text{Tor}_q^A(B, M), -) \Rightarrow \text{Ext}_A^\bullet(M, -)$  constructed in 2.1.4. The edge morphisms from the fiber to the limit in this sequence are then maps  $e : E_2^{\bullet,0}(-) \cong \text{Ext}_B^\bullet(M, -) \rightarrow \text{Ext}_A^\bullet(M, -)$  which we are assuming to be isomorphisms.

Assume that  $r \geq 0$  and that we know that  $\text{Tor}_q^A(B, M) = 0$  if  $0 < q < r$ . Then  $E_{r+1}^{p,q} = 0$  if  $0 < q < r$ ,  $E_{r+1}^{p,q} = E_2^{p,q}$  if  $(p, q) \in \{(r, 0), (0, r), (r+1, 0)\}$ , and convergence implies that we have an exact sequence

$$E_{r+1}^{r,0}(-) \xrightarrow{e} \text{Ext}_A^r(M, -) \longrightarrow E_{r+1}^{0,r}(-) \xrightarrow{d_{r+1}^{0,r}} E_{r+1}^{r+1,0}(-) \xrightarrow{e} \text{Ext}_A^{r+1}(M, -)$$

This tells us that  $E_2^{0,r}(-) = \text{hom}_B(\text{Tor}_r^A(B, M), -)$  vanishes identically on  ${}_B\text{Mod}$  and of course allows us to conclude that  $\text{Tor}_r^A(B, M) = 0$ . This argument clearly gives us  $(b)$  by induction.

The implication  $(b) \Rightarrow (c)$  is immediate. To show that  $(c) \Rightarrow (a)$ , we remark that  $(c)$  implies that the spectral sequence constructed in 2.3.3 degenerates, so that the edge morphisms  $\text{Ext}_{B^e}^\bullet(B, -) \rightarrow \text{Ext}_{A^e}^\bullet(A, -)$  are isomorphisms on  ${}_B\text{Mod}_B$ ;

(a) now follows from the easily established fact that for an algebra  $\Lambda$  there is a natural isomorphism

$$\mathrm{Ext}_{\Lambda}^{\bullet}(-, -) \cong H^{\bullet}(\Lambda, \mathrm{hom}(-, -)) \quad (13)$$

of bifunctors defined on  ${}_{\Lambda}\mathrm{Mod}$ , which is compatible with extension of scalars.

Recall that there is a map  $\top : \mathrm{Tor}_{\bullet}^A(B, B) \otimes \mathrm{Tor}_{\bullet}^{A^{\mathrm{op}}}(B, B) \rightarrow \mathrm{Tor}_{\bullet}^{A^e}(B^e, B^e)$  which is an isomorphism, cf. [5, Theorem XI.3.1]. From this and the obvious existence of an isomorphism  $\mathrm{Tor}_{\bullet}^{A^{\mathrm{op}}}(B, B) \cong \mathrm{Tor}_{\bullet}^A(B, B)$  we see that (c) implies that  $\phi^e : A^e \rightarrow B^e$  itself satisfies (c). Using what we have already proved, we conclude that (c)  $\Rightarrow$  (d).

To show (d)  $\Rightarrow$  (a), assume  $\phi^e$  is a homological epimorphism. Applying the implication (a)  $\Rightarrow$  (c) to  $\phi^e$ , we get that

$$(\mathrm{Tor}_{\bullet}^A(B, B) \otimes \mathrm{Tor}_{\bullet}^{A^{\mathrm{op}}}(B, B))_+ \cong \mathrm{Tor}_{+}^{A^e}(B^e, B^e) = 0,$$

so  $\mathrm{Tor}_{+}^A(B, B) = 0$ . Now we can use the fact that (b)  $\Rightarrow$  (a) to conclude that  $\phi$  is a homological epimorphism.

Finally, the statement about Hochschild cohomology follows from the fact that (c) implies that the spectral sequence in 2.3.3 degenerates and the statement about cup products follows from 2.3.4.  $\square$

**5.4.** The following is an easy corollary of 5.3:

**Corollary.** *Let  $\phi : A \rightarrow B$  be a homological epimorphism. If  $I = \ker \phi$ , then  $\mathrm{Ext}_{A^e}^{\bullet}(I, -)$  vanishes identically on  ${}_B\mathrm{Mod}_B$ .*

*Proof.* Let  $M \in {}_B\mathrm{Mod}_B$  and consider the long exact sequence of the functor  $\mathrm{Ext}_{A^e}^{\bullet}(-, M)$  corresponding to  $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ . The composition of the induced map

$$\mathrm{Ext}_{A^e}^{\bullet}(B, M) \rightarrow \mathrm{Ext}_{A^e}^{\bullet}(A, M)$$

with the isomorphism  $\mathrm{Ext}_{B^e}^{\bullet}(B, M) \rightarrow \mathrm{Ext}_{A^e}^{\bullet}(B, M)$  coincides with the isomorphism (12), so we see that  $\mathrm{Ext}_{A^e}^{\bullet}(I, M) = 0$ .  $\square$

**5.5.** Just as easily, we obtain the following much more interesting result. This is what the first two parts of [26, Proposition 3.2] become when taking 5.3 into account.

**Corollary.** *Let  $\phi : A \rightarrow B$  be a homological epimorphism and let  $I = \ker \phi$ . There is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0(A, I) \longrightarrow HH^0(A) \longrightarrow HH^0(B) \longrightarrow \mathrm{Ext}_{A^e}^1(A, I) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Ext}_{A^e}^p(A, I) \longrightarrow HH^p(A) \longrightarrow HH^p(B) \longrightarrow \mathrm{Ext}_{A^e}^{p+1}(A, I) \longrightarrow \cdots \end{aligned}$$

The maps  $HH^p(A) \rightarrow HH^p(B)$  appearing in this sequence can be collected into an algebra morphism  $HH^{\bullet}(A) \rightarrow HH^{\bullet}(B)$ .

*Proof.* This is just the long exact sequence for the functor  $\mathrm{hom}_{A^e}(A, -)$  corresponding to the short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

up to the isomorphism  $HH^{\bullet}(B) \cong H^{\bullet}(A, B)$  coming from (12).

The last statement is a direct consequence of the last statement in 5.3.  $\square$

**5.6.** If  $\phi : A \rightarrow B$  is an epimorphism and  $I = \ker \phi$ ,  $\mathrm{Tor}_1^A(B, B) \cong I/I^2$ , so 5.3 implies that a homological ideal is idempotent. We have the following partial converse:

**Proposition.** *Let  $I \triangleleft A$  be flat as an  $A$ -module on the left or on the right and put  $B = A/I$ . If it is idempotent, then it is homological. In general,  $H^0(B, -) \cong H^0(A, -)$  and there is a long exact sequence*

$$\begin{aligned} 0 \rightarrow H^1(B, M) \rightarrow H^1(A, M) \rightarrow \text{Ext}_{A^e}^0(I/I^2, M) \xrightarrow{\sim \zeta} H^2(B, M) \rightarrow \cdots \\ \cdots \rightarrow H^p(B, M) \rightarrow H^p(A, M) \rightarrow \text{Ext}_{A^e}^{p-1}(I/I^2, M) \xrightarrow{\sim \zeta} H^{p+1}(B, M) \rightarrow \cdots \end{aligned}$$

functorial on  $B$ -bimodules  $M$ . Here  $\zeta \in H^2(B, I/I^2)$  is the class of the singular extension

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0$$

In particular, restriction of scalars induces functorial isomorphisms

$$H^p(B, -) \rightarrow H^p(A, -)$$

on  $B$ -bimodules for  $p > \text{pdim}_{A^e} I/I^2 + 1$ .

*Proof.* Looking at the long exact sequence for  $\text{Tor}_\bullet^A(B, -)$  corresponding to

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

we see that  $\text{Tor}_p^A(B, B) \cong \text{Tor}_{p-1}^A(B, I) = 0$  for  $p > 1$ . Using this together with the convergence of the spectral sequence in 2.3.3 we see that the long exact sequence in the statement exists. All other claims follow at once.  $\square$

**5.7.** We now describe a nice example where one can see 5.6 in nature. Let  $X$  be a finite poset and let  $Y \subset X$  be an order ideal, that is, a subset such that

$$x \in X, y \in Y, x \leq y \implies x \in Y.$$

Let  $A = kX$  and  $kY$  be the incidence algebras of  $X$  and  $Y$ —recall that  $kX$ , for example, can be seen as the quotient of the path algebra on the quiver given by the Hasse diagram of  $X$  divided by the ideal of all commutativity relations. For simplicity, we identify  $X$  with (the vertex set of) its Hasse diagram.

Let us put  $e = \sum_{x \in X \setminus Y} x$ . This is an idempotent in  $A$ , so  $I_Y = AeA$  is an idempotent ideal. Now,  $I_Y$  is clearly linearly spanned by all paths in  $X$  which go through a vertex in  $X \setminus Y$  and, because  $Y$  is an order ideal, these are precisely the paths in  $X$  which *start* at a vertex of  $X \setminus Y$ . In other words,  $I_Y = Ae$ . In particular,  $I_Y$  is projective as a left  $A$ -module and 5.6 tells us that it is a homological ideal. Since  $kX/I_Y \cong kY$ , the long exact sequence of 5.5 is then

$$\begin{aligned} 0 \rightarrow H^0(kX, I_Y) \rightarrow HH^0(kX) \rightarrow HH^0(kY) \rightarrow \text{Ext}_{(kX)^e}^1(kX, I_Y) \rightarrow \cdots \\ \cdots \rightarrow \text{Ext}_{(kX)^e}^p(kX, I_Y) \rightarrow HH^p(kX) \rightarrow HH^p(kY) \rightarrow \text{Ext}_{(kX)^e}^{p+1}(kX, I_Y) \rightarrow \cdots \end{aligned} \tag{14}$$

A well known result of Gerstenhaber and Schack [14] states that  $HH^\bullet(kX)$  is canonically isomorphic to the simplicial cohomology of the geometric realization  $|X|$  of  $X$  and, of course, a similar statement holds for  $kY$ . Using the technique of [14], one can easily show that  $\text{Ext}_{(kX)^e}^\bullet(kX, I_Y) \cong H^\bullet(|X|, |Y|)$ , the simplicial cohomology of the pair  $(|X|, |Y|)$ . Moreover, under these isomorphisms the long exact sequence (14) corresponds to the long exact sequence for the cohomology of the pair  $(|X|, |Y|)$ .

More generally, let  $X$  be a finite poset as before and let now  $Y \subset X$  be an arbitrary subset. Let  $\text{Ch}(X)$  and  $\text{Ch}(Y)$  be the posets of chains of  $X$  and  $Y$ , respectively. Then  $\text{Ch}(Y)$  is an order ideal in  $\text{Ch}(X)$ . Recalling that the simplicial complex which realizes  $\text{Ch}(X)$  is the barycentric subdivision of the one which

realizes  $X$  and that simplicial cohomology is invariant under such subdivisions, we see that  $HH^\bullet(k\text{Ch}(X)) \cong HH^\bullet(kX)$ . Up to these isomorphisms, the long exact sequence (14) provides a long exact sequence relating  $HH^\bullet(kX)$ ,  $HH^\bullet(kY)$  and

$$\text{Ext}_{(k\text{Ch}(X))^e}^\bullet(k\text{Ch}(X), I_{\text{Ch}(Y)}),$$

which again is isomorphic to a relative cohomology group. We remark that it would be useful to have a description of these cohomology groups directly in terms of  $kX$  and  $kY$ .

In any case, we see that the long exact sequences for simplicial cohomology of pairs of finite simplicial complexes are all special cases of 5.5.

**5.8.** As observed in 5.6, if  $\phi : A \rightarrow B$  is a homological epimorphism, the idempotency of  $I = \ker \phi$  follows from the vanishing of  $\text{Tor}_1^A(B, B)$ . Looking at what happens in degree two we find the following lemma:

**Lemma.** [1, Lemma 1.4] *If  $\phi : A \rightarrow B$  is a homological epimorphism and  $I = \ker \phi$ , then the multiplication map  $I \otimes_A I \rightarrow I$  is an isomorphism.*  $\square$

Thus a homological ideal is idempotent as a bimodule.

**5.9.** One can give various kinds of combinatorial conditions on ideals of algebras given by quivers and relations that ensure that they are homological. We give as a simple instance a partial converse of 5.8:

**Proposition.** *Let  $Q$  be a finite quiver with path algebra  $kQ$ , let  $J \triangleleft kQ$  be an admissible ideal and put  $A = kQ/J$ . Let  $e \in Q_0$  be a vertex such that there are no oriented circuits in  $Q$  starting in  $e$ . Put  $I = AeA$  and  $B = A/I$ . Then  $I$  is a homological ideal iff multiplication gives an isomorphism  $\mu : I \otimes_A I \rightarrow I$ .*

*Proof.* Necessity follows from 5.8. In order to show the sufficiency of the condition we need only show that  $\text{Tor}_+^A(B, B) = 0$ . Using twice the long exact sequences for the functor  $\text{Tor}_\bullet^A$  corresponding to the short exact sequence of  $A$ -bimodules  $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ , we see that

$$\text{Tor}_p^A(B, B) \cong \begin{cases} B, & \text{if } p = 0; \\ I/I^2, & \text{if } p = 1; \\ \text{Tor}_1^A(B, I), & \text{if } p = 2; \\ \text{Tor}_{p-2}^A(I, I), & \text{if } p \geq 3. \end{cases}$$

and that there is an exact sequence

$$0 \longrightarrow \text{Tor}_1^A(B, I) \longrightarrow I \otimes_A I \xrightarrow{\mu} I \longrightarrow 0$$

The idempotency of  $I$  and the hypothesis on  $\mu$  imply then that we will be done if we show that  $\text{Tor}_+^A(I, I) = 0$ .

Let  $E$  be the subalgebra of  $A$  generated by the vertices and let  $\mathfrak{r} = \text{rad } A$  be the Jacobson radical. Claude Cibils has shown in [7] that  $A$  has a projective resolution as a  $A$ -bimodule of the form

$$\begin{aligned} \cdots \longrightarrow A \otimes_E \mathfrak{r}^{\otimes_E p} \otimes_E A \longrightarrow A \otimes_E \mathfrak{r}^{\otimes_E (p-1)} \otimes_E A \longrightarrow \cdots \\ \cdots \longrightarrow A \otimes_E \mathfrak{r}^{\otimes_E 2} \otimes_E A \longrightarrow A \otimes_E \mathfrak{r} \otimes_E A \longrightarrow A \otimes_E A \longrightarrow A \longrightarrow 0 \end{aligned}$$

It follows that  $\text{Tor}_\bullet^A(I, I)$  is the homology of a complex which for each  $p \geq 0$  has degree  $p$  component given by  $I \otimes_E \mathfrak{r}^{\otimes_E p} \otimes_E I$ .

Now, since  $\mathfrak{r}$  is the ideal generated by the arrows of  $Q$  and there are no oriented circuits in  $Q$  based at  $e$ , it is clear that  $I \otimes_E \mathfrak{r}^{\otimes_E p} \otimes_E I = 0$  if  $p > 0$ . The homology

of the complex in question is thus trivially computable and we see at once that  $\text{Tor}_+^A(I, I) = 0$ .  $\square$

**5.10.** The following proposition is essentially [26, Proposition 3.2.(c)] except that we do not assume that the ideal  $I$  is homological. The proof is in fact exactly the same as the one given there, but we include it for completeness.

**Proposition.** *Let  $A$  be a finite dimensional  $k$ -algebra and let  $e \in A$  be an idempotent. Put  $I = AeA$  and assume the multiplication in  $A$  induces an isomorphism of  $A$ -bimodules  $Ae \otimes eA \rightarrow AeA = I$ . Let  $D(-) = \text{hom}_k(-, k)$  be the usual duality and let  $B = A/I$ . Then  $I$  is a homological ideal and there is a long exact sequence*

$$\begin{aligned} 0 \longrightarrow Z(A) \cap I \longrightarrow HH^0(A) \longrightarrow HH^0(B) \longrightarrow \text{Ext}_A^1(D(eA), Ae) \longrightarrow \cdots \\ \cdots \longrightarrow \text{Ext}_A^p(D(eA), Ae) \longrightarrow HH^p(A) \longrightarrow HH^p(B) \longrightarrow \text{Ext}_A^{p+1}(D(eA), Ae) \longrightarrow \cdots \end{aligned}$$

Notice that the first claim is a generalization of 5.9.

*Proof.* The hypothesis on  $e$  implies that  $I$  is idempotent and projective, so it is homological by 5.6. The long exact sequence whose existence is claimed in the statement is the one from 5.5 up to identifications.

First of all,  $H^0(A, I) \cong \text{hom}_{Ae}(A, I) \cong Z(A) \cap I$ , so this takes care of the beginning of the sequence. Next, dualizing the isomorphism  $I \cong Ae \otimes eA$ , we see that

$$D(I) \cong D(eA) \otimes D(Ae), \quad (15)$$

so we have a chain of isomorphisms

$$\begin{aligned} D(\text{Ext}_{Ae}^p(A, I)) &\cong \text{Tor}_p^{Ae}(A, D(I)) && \text{by [5, IX, ex. 8]} \\ &\cong \text{Tor}_p^{Ae}(A, D(eA) \otimes D(Ae)) && \text{by (15)} \\ &\cong \text{Tor}_p^A(D(Ae), D(eA)) && \text{by [5, Corol. IX.4.4]} \\ &\cong D(\text{Ext}_A^p(D(eA), Ae)) && \text{by [5, Prop. VI.5.3].} \end{aligned}$$

Dualizing again, we see that  $\text{Ext}_{Ae}^p(A, I) \cong \text{Ext}_A^p(D(eA), Ae)$ .  $\square$

**5.11.** From this proposition one can easily obtain the long exact sequence that Dieter Happel constructed in [17] for a one-point extension of a finite dimensional algebra. Indeed, let  $B$  be a finite dimensional  $k$ -algebra, let  $M \in {}_B\text{mod}$  be non-zero and consider the matrix algebra

$$A = \begin{pmatrix} B & M \\ 0 & k \end{pmatrix}.$$

If  $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , then one easily sees that  $I = AeA$  satisfies the hypothesis of 5.10; moreover, it is clear that  $A/I \cong B$ . Consider now the long exact sequence corresponding to the obvious short exact sequence

$$0 \longrightarrow M \longrightarrow Ae \longrightarrow D(eA) \longrightarrow 0$$

and the functor  $\text{hom}_A(-, Ae)$ .

First, if  $p \geq 2$ , we have a portion of that sequence that reads

$$\begin{aligned} \cdots \longrightarrow \text{Ext}_A^{p-1}(Ae, Ae) \longrightarrow \text{Ext}_A^{p-1}(M, Ae) \longrightarrow \\ \longrightarrow \text{Ext}_A^p(D(eA), Ae) \longrightarrow \text{Ext}_A^p(Ae, Ae) \longrightarrow \cdots \end{aligned}$$

so, since  $Ae$  is projective, we conclude that  $\text{Ext}_A^p(D(eA), Ae) \cong \text{Ext}_A^{p-1}(M, Ae)$ .



Second, the beginning of that long exact sequence is

$$\begin{aligned} 0 \longrightarrow \text{hom}_A(D(eA), Ae) \longrightarrow \text{hom}_A(Ae, Ae) \longrightarrow \\ \longrightarrow \text{hom}_A(M, Ae) \longrightarrow \text{Ext}_A^1(D(eA), Ae) \longrightarrow 0 \end{aligned}$$

Using this and the fact that  $\text{hom}_A(D(eA), Ae) = 0$  and  $\text{hom}_A(Ae, Ae) \cong eAe \cong k$ , we see that  $\text{Ext}_A^1(D(eA), Ae) \cong \text{hom}_A(M, Ae)/k$ .

Finally, since  ${}_B\text{mod}$  is a convex subcategory of  ${}_A\text{mod}$  and  $\text{hom}_A(B, Ae) \cong M$ , there is an isomorphism  $\text{Ext}_A^\bullet(M, Ae) \cong \text{Ext}_B^\bullet(M, M)$ .

Using the isomorphisms thus obtained and the fact that  $Z(A) \cap I = 0$ , the long exact sequence in 5.10 becomes

$$\begin{aligned} 0 \longrightarrow HH^0(A) \longrightarrow HH^0(B) \longrightarrow \text{Ext}_B^1(M, M)/k \longrightarrow \cdots \\ \cdots \longrightarrow \text{Ext}_B^p(M, M) \longrightarrow HH^p(A) \longrightarrow HH^p(B) \longrightarrow \text{Ext}_B^{p+1}(M, M) \longrightarrow \cdots \end{aligned}$$

which is, precisely, Happel's long exact sequence. This derivation is explained in [26].

**5.12.** The long exact sequence of Happel allows us to obtain information on the Hochschild cohomology of an algebra which can be built in steps from a simpler one by doing one-point extensions and coextensions. Indeed, in terms of algebras given by quivers and relations, one-point extensions correspond to the process of adding a new vertex 'at the top' of the quiver — dually, one point coextensions correspond to adding a new vertex 'at the bottom' — and sufficiently simple quivers can be constructed inductively starting from a vertex by adding new vertices both on the top and on the bottom. See [8, 9, 10] for examples on how this inductive procedure for the computation of cohomology is carried out.

From this point of view 5.10 becomes interesting: it allows us to add vertices 'in the middle' in certain situations, enlarging the scope for such inductive calculations. The following proposition is a very simple example of this.

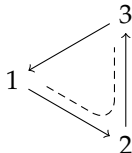
**5.13.** If  $Q$  is a quiver,  $e \in Q_0$  a vertex and  $\gamma = \alpha_1 \cdots \alpha_k$  a path in  $Q$ , we say that  $\gamma$  has  $e$  as an internal vertex if  $e$  is either the source of one of  $\alpha_1, \dots, \alpha_{k-1}$  or the target of one of  $\alpha_2, \dots, \alpha_k$ .

**Proposition.** Let  $Q$  be a finite quiver with path algebra  $kQ$ , let  $J \triangleleft kQ$  be an admissible monomial ideal and put  $A = kQ/J$ . Let  $e \in Q_0$  be a vertex in  $Q$  such that no minimal generator of  $J$  has  $e$  as an internal vertex. Then  $I = AeA$  is a homological ideal and 5.10 provides a long exact sequence relating  $HH^\bullet(A)$  and  $HH^\bullet(A/I)$ .

In fact, it is not difficult to show that the condition on  $e$  given in the proposition is actually necessary for the ideal  $I$  to be homological in this case.

*Proof.* This follows from 5.10 if we can prove that multiplication induces an isomorphism  $Ae \otimes eA \rightarrow AeA$ . The hypothesis on  $e$  is precisely what is needed for this.  $\square$

**5.14.** As a toy example of how one can use proposition 5.13, consider the algebra  $A$  obtained as the quotient of the path algebra of the following quiver by the ideal generated by the dotted path.



It is clear that the idempotent  $e_1$  corresponding to the vertex 1 satisfies the condition of the proposition, so the ideal  $I = Ae_1A$  is homological. The algebra  $B = A/I$  is the path algebra of the convex subquiver spanned by the vertices 2 and 3, which is a tree, so  $HH^\bullet(B) \cong k$ . Now, the  $A$ -module  $D(e_1A)$  has a projective resolution

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_2 \longrightarrow D(e_1A) \longrightarrow 0$$

where  $P_i$  is the indecomposable projective module corresponding to the vertex  $i$ , the morphism  $P_2 \rightarrow P_2$  maps the top onto the socle of  $P_2$  and the morphism  $P_3 \rightarrow P_2$  is the inclusion. Applying the functor  $\text{hom}_A(-, Ae_1)$  and computing, one easily concludes that

$$\text{Ext}_A^p(D(e_1A), Ae_1) \cong \begin{cases} k & \text{if } p = 0 \text{ or } p = 1; \\ 0 & \text{if } p \geq 2. \end{cases}$$

The long exact sequence of 5.10 then reduces in this case to isomorphisms  $HH^p(A) = 0$  for all  $p \geq 2$  and an exact sequence

$$\begin{aligned} 0 \longrightarrow Z(A) \cap I \longrightarrow HH^0(A) \longrightarrow HH^0(B) \longrightarrow \\ \longrightarrow \text{Ext}_A^1(D(e_1A), Ae_1) \longrightarrow HH^1(A) \longrightarrow 0 \end{aligned}$$

The map  $HH^0(A) \rightarrow HH^0(B)$  is surjective, so we see that  $HH^0(A) \cong k^2$  and  $HH^1(A) \cong \text{Ext}_A^1(D(e_1A), Ae_1) \cong k$ .

Notice that  $A$  is not a one-point (co)extension, so one cannot use the classical Happel long exact sequence to compute  $HH^\bullet(A)$ .

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